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Constrained inference in linear regression

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ABSTRACT

Regression analysis is probably one of the most used statistical techniques. We consider the case when the regression function is monotonically changing with some or all of the predictors in a region of interest. Restricted confidence interval for the mean of the regression function is constructed when two predictors are present. Earlier analyses would allow an investigator either to ignore monotonicity altogether or to consider only one predictor at a time but not both simultaneously. The methodologies developed are applied on a real data set to study the effects of patients' age and infection risk on their length of stay in US hospitals.

1. Introduction

1.1. Preliminaries

Consider the standard linear regression model $\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where $\boldsymbol{Y}$ is an $(n \times 1)$ vector, $\boldsymbol{X}$ is an $(n \times p)$ matrix of rank p, $\boldsymbol{\beta}$ is a $(p \times 1)$ vector of unknown parameters, and $\boldsymbol{\epsilon}$ is an $(n \times 1)$ multivariate normal vector of errors with zero mean and covariance matrix $\sigma^2\boldsymbol{I}$. There is a wide range of applications where the sign constraints on regression coefficients are useful. This area of statistical research is known as *non-negative least squares (NNLS)*. In image processing or spectral analysis NNLS is quite well-known, where the signs of the regression parameters can be estimated, or known a priori [2,4,5,7,8,19,23]. NNLS regression can be a useful tool for matrix factorization [10]. The non-negative Garrote [3] uses a sign-constraint, where the signs are derived from an initial estimator as is the positive Lasso [6]. This constraint is particularly relevant when modeling non-negative data, which emerge, e.g., in the form of pixel intensity values of an image, time measurements, histograms or count data, economical quantities such as prices, incomes and growth rates. Non-negativity constraints occur naturally in numerous deconvolution and unmixing problems in diverse fields such as acoustics [14], astronomical imaging [1], genomics [13], proteomics [21], spectroscopy [5] and network tomography [15]; see [4] for a survey.

It is more common in order-restricted regression analysis to consider inference under null hypothesis of the type $\boldsymbol{R\beta} = \boldsymbol{r}$ versus $\boldsymbol{R\beta} \geq \boldsymbol{r}$, $\boldsymbol{R\beta} \neq \boldsymbol{r}$, for some matrix $\boldsymbol{R}$, vector $\boldsymbol{r}$ [18,20]. Restricted statistical inference in regression analysis under nonnegativity constraints on $\boldsymbol{\beta}$ (NNLS) is rare at best. This emerges when the experimenter believes that the regression

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function changes monotonically with the predictors (see references above). [16] considered the inference for the mean of the response variable when one predictor variable is present, but their work does not extend to higher dimensional cases in a straightforward manner. In this paper we consider the case of *two* predictors following the same format as theirs. Increasing the number of predictors not only makes practically more useful results but also generates new spaces in null hypothesis parameter region which has no counterpart in lower dimensions (e.g., mixed signs in Section 4). We have used tools from calculus and geometry [9] in our analysis. Graphs are used throughout the paper for illustration, where we have used the convention that arrows on axes indicate to the positive directions. Often we use a (cross-sectional) two-dimensional graph to illustrate a three-dimensional region for clarity or when the three dimensional graph is messy to display.

To obtain confidence intervals we have considered the acceptance regions of corresponding one-sided tests [12,23]. Least favorable distributions are used for calculating the critical values of the tests, however, these distributions are known to be conservative. Restricted likelihood ratio tests (LRT) are used, but it is shown that often these tests perform poorly than a related unrestricted test. In such cases, we have proposed an ad hoc test in similar spirit as in [16] to improve on the LRT.

We have applied our methodology on the SENIC data [11]. The primary objective of the study was to determine whether infection surveillance and control programs have reduced the rates of nosocomial (hospital-acquired) infection in US hospitals. Here we suspect β_1 to be positive and β_2 to be negative because older patients seem to stay longer in hospital and higher infection is associated with shorter hospital stay. Whereas the ordinary regression analysis would ignore this monotonicity information, our analysis implements it. Following [16] one has to consider these important predictors only *one at a time*. Our analysis enables one to consider them *simultaneously*. See Section 7 for data analysis on the example.

1.2. Regression basics

Assuming the first column of $\boldsymbol{X}$ to be all ones, and for two predictor variables X_1, X_2, for a sample of size n, the regression model becomes, $Y_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \epsilon_i,\ 1 \le i \le n$. Let $\hat\beta_0, \hat\beta_1$ and $\hat\beta_2$ be the unrestricted maximum likelihood estimates (MLEs) of β_0, β_1 and β_2 respectively. Let, $S^2_{x_1} = \sum x^2_{1i},\ S^2_{x_2} = \sum x^2_{2i}$ and $S^2 = \sum (Y_i - \hat\beta_0 - \hat\beta_1 x_{1i} - \hat\beta_2 x_{2i})^2/\nu$, where $\nu = n-3$. We assume that the columns of $\boldsymbol{X}$ are orthogonal, that is, $\sum_i x_{1i} = 0,\ \ \sum_i x_{2i} = 0$ and $\sum_i x_{1i}x_{2i} = 0$.

Then it is well known that $\hat\beta_0, \hat\beta_1, \hat\beta_2, S^2$ are mutually independent. Further, $\hat\beta_0 \sim \mathcal{N}(\beta_0,\ \sigma^2/n)$, $\hat\beta_1 \sim \mathcal{N}(\beta_1, \sigma^2/S^2_{x_1})$, $\hat\beta_2 \sim \mathcal{N}(\beta_2, \sigma^2/S^2_{x_2})$ and $\nu S^2/\sigma^2 \sim \chi^2_\nu$.

Let $\boldsymbol\gamma = (\gamma_0, \gamma_1, \gamma_2)^\top$ where $\gamma_0 = \sqrt{n}\beta_0$, $\gamma_1 = S_{x_1}\beta_1$, $\gamma_2 = S_{x_2}\beta_2$ then the unrestricted MLE of $\boldsymbol\gamma$ is $\hat{\boldsymbol\gamma} = (\hat\gamma_0, \hat\gamma_1, \hat\gamma_2)^\top = (\sqrt{n}\hat\beta_0, S_{x_1}\hat\beta_1, S_{x_2}\hat\beta_2)^\top \sim \boldsymbol{\mathcal{N}}_3(\boldsymbol\gamma, \sigma^2\boldsymbol{I})$.

Under the constraints $\beta_1 \ge 0, \beta_2 \ge 0$, the restricted MLEs of β_i's are given by, $\beta_0^* = \hat\beta_0$, $\beta_1^* = \max\{\hat\beta_1, 0\} = \beta_1^+$, $\beta_2^* = \max\{\hat\beta_2, 0\} = \beta_2^+$. Then the restricted parameter space for $\boldsymbol\gamma$ is $\{\boldsymbol\gamma : \gamma_0 \in \mathbb{R}, \gamma_1 \ge 0, \gamma_2 \ge 0\}$. The restricted MLEs of $\boldsymbol\gamma$ are $\gamma_0^* = \hat\gamma_0$, $\gamma_1^* = \max\{\hat\gamma_1, 0\} = \gamma_1^+$, $\gamma_2^* = \max\{\hat\gamma_2, 0\} = \gamma_2^+$.

The case of σ^2 *known* is considered in Sections 2–5. Section 6 considers σ^2 *unknown* case. We end with some discussion in Section 8. Statistical inference under other combinations of sign restrictions of β_1, β_2 can also be developed similarly. Supplement of this paper contains Lemmas 1–4 with proofs, graphs S1–S3, a chart summarizing the distributions of LRT in limiting cases of (x_{01}, x_{02}), tables of critical values and formulas of confidence intervals in original variables (see [17] for further details). The computer programs needed for the example and calculation of critical values are written in fortran and R (available from the authors on request).

2. Inferences for $\boldsymbol{\beta_0 + \beta_1 x_{01} + \beta_2 x_{02}}$

We consider inferences about the mean function $\mathrm{E}(Y) = \beta_0 + \beta_1 x_{01} + \beta_2 x_{02}$ at predictor variable values (x_{01}, x_{02}) for different possible signs of x_{01} and x_{02}.

2.1. Test for $\beta_0 + \beta_1 x_{01} + \beta_2 x_{02}$ $(x_{01} > 0, x_{02} > 0)$

First we consider the hypotheses,

$$\boldsymbol{G}_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \le l, \quad \beta_1 \ge 0,\ \beta_2 \ge 0, \qquad \boldsymbol{G}_1 : \beta_1 \ge 0,\ \beta_2 \ge 0, \tag{2.1}$$

for some $l \in \mathbb{R}$. Using the transformation from $\boldsymbol\beta$ to $\boldsymbol\gamma$, the constraint $\beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \le l$ in (2.1) becomes, $\frac{\gamma_0}{\sqrt{n}} + \frac{\gamma_1 x_{01}}{S_{x_1}} + \frac{\gamma_2 x_{02}}{S_{x_2}} \le l$, or, $\gamma_2 \le b_1 - c_1\gamma_0 - d_1\gamma_1$, where $b_1 = \frac{l S_{x_2}}{x_{02}}$, $c_1 = \frac{S_{x_2}}{x_{02}\sqrt{n}}$ and $d_1 = \frac{x_{01} S_{x_2}}{x_{02} S_{x_1}}$.

Then, using γ_i hypotheses (2.1) are,

$$\boldsymbol{G}_{01} : 0 \le \gamma_2 \le b_1 - c_1\gamma_0 - d_1\gamma_1, \quad 0 \le \gamma_1, \qquad \boldsymbol{G}_{11} : \gamma_1 \ge 0,\ \gamma_2 \ge 0, \tag{2.2}$$

respectively. To visualize geometrically the sets $\boldsymbol{G}_{01}$ and $\boldsymbol{G}_{11}$ in the $\boldsymbol\gamma$ space, let $\boldsymbol{K}$ be the closed convex cone bounded by the hyperplanes $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = 0, \gamma_1 \ge 0, \gamma_2 \ge 0\}$, $\{\gamma_2 = 0, 0 \le \gamma_1 \le \frac{-c_1\gamma_0}{d_1}, \gamma_0 \le 0\}$, and $\{\gamma_1 = 0, 0 \le \gamma_2 \le -c_1\gamma_0, \gamma_0 \le 0\}$ and let $\boldsymbol{L} = (b_1/c_1, 0, 0)$, then $\boldsymbol{G}_{01}$ is the shifted cone $\boldsymbol{K} + \boldsymbol{L}$. Shifting the cone $\boldsymbol{K}$ by b_1/c_1 units along the positive direction

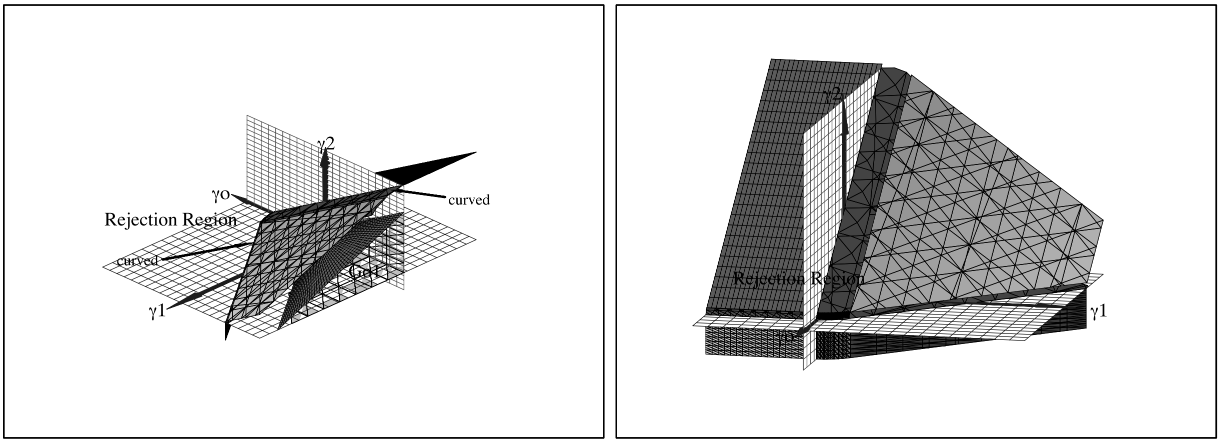

Fig. 1. Left: The region $\boldsymbol{G}_{01}$ and the rejection region of LRT of (2.2). A look from the negative direction of γ_0 axis. Right: Another view of the rejection region of LRT of (2.2) from the positive direction of γ_0 axis. Here region $\boldsymbol{G}_{01}$ is hidden behind.

of the γ_0 axis, we get the faces of $\boldsymbol{G}_{01}$ as $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1, \gamma_1 \geq 0, \gamma_2 \geq 0\}$, $\{\gamma_2 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, \gamma_0 \leq \frac{b_1}{c_1}\}$, and $\{\gamma_1 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, \gamma_0 \leq \frac{b_1}{c_1}\}$ (see Fig. 1 left).

The dual cone of $\boldsymbol{G}_{01}$ is $G_{01}^* = \boldsymbol{K}^* + \boldsymbol{L}$, where $\boldsymbol{K}^*$ is the Fenchel dual cone of $\boldsymbol{K}$, consisting of all the vectors that make obtuse angles with all the vectors in $\boldsymbol{K}$ [19], so that $\hat{\boldsymbol{\gamma}} \in G_{01}^*$ implies that the projection of $\hat{\boldsymbol{\gamma}}$ onto $\boldsymbol{G}_{01}$ is $\boldsymbol{L}$. The dual cone $\boldsymbol{K}^*$ is polyhedral with faces $\{\gamma_0 - c_1\gamma_2 = 0, \gamma_0 \geq 0, \gamma_1 \leq (d_1/c_1)\gamma_0\}$, $\{\gamma_0 - (c_1/d_1)\gamma_1 = 0, \gamma_0 \geq 0, \gamma_2 \leq (1/c_1)\gamma_0\}$ and $\{\gamma_0 = 0, \gamma_1 \leq (d_1/c_1)\gamma_0, \gamma_2 \leq (1/c_1)\gamma_0\}$ (Lemma 1 in the Supplement). The faces of the dual cone are useful to develop the rejection region.

The restricted MLE $\boldsymbol{\gamma}^*$ of $\boldsymbol{\gamma}$ under $\boldsymbol{G}_{11}$ is $(\gamma_0^*, \gamma_1^*, \gamma_2^*)^\top = (\hat{\gamma}_0, \hat{\gamma}_1^+, \hat{\gamma}_2^+)^\top$. Let $\bar{\boldsymbol{\gamma}}$ denote the MLE of $\boldsymbol{\gamma}$ under $\boldsymbol{G}_{01}$. Then $\bar{\boldsymbol{\gamma}}$ is the equal weight projection of $\hat{\boldsymbol{\gamma}}$ onto $\boldsymbol{G}_{01}$. Note that in this problem, $\bar{\boldsymbol{\gamma}}$ is also the equal-weight projection of $\boldsymbol{\gamma}^*$ onto $\boldsymbol{G}_{01}$, and $\hat{\boldsymbol{\gamma}} \in \boldsymbol{G}_{01}^*$ implies that $\bar{\boldsymbol{\gamma}} = \boldsymbol{L} = (b_1/c_1, 0, 0)^\top$.

For testing $\boldsymbol{G}_{01}$ versus $\boldsymbol{G}_{11} - \boldsymbol{G}_{01}$, the LRT rejects $\boldsymbol{G}_{01}$ for large values of the test statistic,

$$\bar{\chi}_{01}^2 \equiv -2\log \boldsymbol{\Lambda} = (\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2)/\sigma^2 = \|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2/\sigma^2, \tag{2.3}$$

where $\boldsymbol{\Lambda}$ is the appropriate LRT statistic. Next we investigate the rejection region of LRT in (2.3), two different views of which are shown in Fig. 1.

Let $\{\hat{\boldsymbol{\gamma}} : \|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\| > C_\alpha\sigma\}$ be the rejection region for a α-level test for some critical value C_α. For ease of computation, depending on the signs of $\hat{\gamma}_i$, $i = 1, 2$ and hyperplanes of interest, we divide the $\mathbb{R}^3$ space into thirteen *disjoint* polyhedral cone regions and calculate the test statistic $\bar{\chi}_{01}^2$ in (2.3) for each region separately. Finally we combine all these regions to yield the nine *disjoint* regions as stated in (2.6).

First consider when $\hat{\boldsymbol{\gamma}} \in \{(\gamma_0, \gamma_1, \gamma_2) : \gamma_1 < 0, \gamma_2 < 0\} = \boldsymbol{S}_1 \uplus \boldsymbol{S}_2$, where $\uplus$ means disjoint union, $\boldsymbol{S}_1 = \{\gamma_0 < \frac{b_1}{c_1}, \gamma_1 < 0, \gamma_2 < 0\}$ and $\boldsymbol{S}_2 = \{\gamma_0 \geq \frac{b_1}{c_1}, \gamma_1 < 0, \gamma_2 < 0\}$.

From (2.3), when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_1$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\| = \|(\hat{\gamma}_0, 0, 0) - (\hat{\gamma}_0, 0, 0)\| = 0$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_2$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\| = \|(\hat{\gamma}_0, 0, 0) - (\frac{b_1}{c_1}, 0, 0)\| = \hat{\gamma}_0 - \frac{b_1}{c_1} \geq C_\alpha\sigma$ and hence the boundary of the rejection region in $\boldsymbol{S}_2$ is $\hat{\gamma}_0 = \frac{b_1}{c_1} + C_\alpha\sigma$ (two dimensional views of this boundary plane are the line *CD* in Fig. 2 (left) when $\gamma_1 = 0$ and the line *CG* in Fig. 2 (right) when $\gamma_2 = 0$).

The plane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ (see (2.2)) intersects the $\gamma_0\gamma_2$-plane (i.e. $\gamma_1 = 0$) at the line $c_1\gamma_0 + \gamma_2 = b_1$ (line *ML* in Fig. 2 (left)). Perpendicular to this line at the point $\boldsymbol{L}$ on the plane $\gamma_1 = 0$ is the line $\gamma_0 - c_1\gamma_2 = \frac{b_1}{c_1}$ (line *NL* in Fig. 2 (left)). The hyperplanes in $\mathbb{R}^3$ generated from the lines $c_1\gamma_0 + \gamma_2 = b_1$ and $\gamma_0 - c_1\gamma_2 = \frac{b_1}{c_1}$ are used to define the regions $\boldsymbol{S}_3, \boldsymbol{S}_4, \boldsymbol{S}_5$ below.

Consider when $\hat{\boldsymbol{\gamma}} \in \{\boldsymbol{\gamma} : \gamma_1 < 0, \gamma_2 \geq 0\} = \boldsymbol{S}_3 \uplus \boldsymbol{S}_4 \uplus \boldsymbol{S}_5$, where $\boldsymbol{S}_3 = \{\gamma_1 < 0, 0 \leq \gamma_2 < b_1 - c_1\gamma_0\}$, $\boldsymbol{S}_4 = \{\gamma_1 < 0, \gamma_2 \geq \max\{b_1 - c_1\gamma_0, \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}\}$ and $\boldsymbol{S}_5 = \{\gamma_1 < 0, 0 \leq \gamma_2 < \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}$, then $\boldsymbol{\gamma}^*$ is the projection of $\hat{\boldsymbol{\gamma}}$ onto the $\gamma_0\gamma_2$ plane and $\bar{\boldsymbol{\gamma}}$ is the projection of $\boldsymbol{\gamma}^*$ onto the edge of $\boldsymbol{G}_{01}$ on the plane $\gamma_0\gamma_2$ (onto the line $c_1\gamma_0 + \gamma_2 = b$), if $\boldsymbol{\gamma}^* \notin \boldsymbol{G}_{01}$.

So when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_3$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\| = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - (\hat{\gamma}_0, 0, \hat{\gamma}_2)\|^2 = 0$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_4$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - ((\hat{\gamma}_0, 0, \hat{\gamma}_2) \cdot \boldsymbol{u})\boldsymbol{u}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{u}$ is a unit vector along the line $c_1\gamma_0 + \gamma_2 = b_1$ on the $\gamma_0\gamma_2$-hyperplane. This gives the hyperplane which is parallel and has $C_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + \gamma_2 = b_1$. The equation of this hyperplane is $c_1\gamma_0 + \gamma_2 = b_1 + \sqrt{1 + c_1^2}C_\alpha\sigma$. So the boundary of the rejection region in $\boldsymbol{S}_4$ is $c_1\gamma_0 + \gamma_2 = b_1 + \sqrt{1 + c_1^2}C_\alpha\sigma$ (line AB in Fig. 2 (left) is the two dimensional view when $\gamma_1 = 0$ of the boundary of the rejection region).

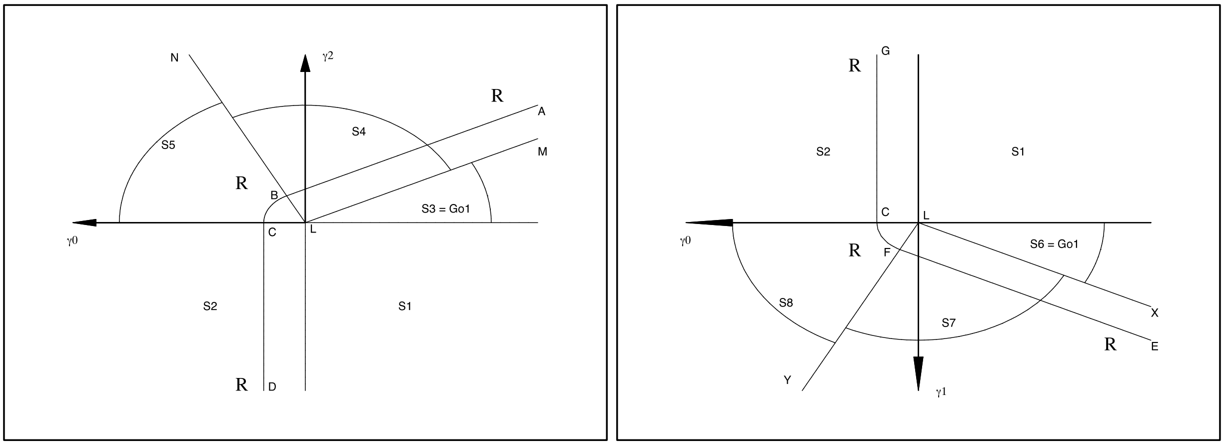

Fig. 2. For $C = (\frac{b_1}{c_1} + C_\alpha\sigma, 0, 0)$, $B = (\frac{b_1}{c_1} + \frac{c_1}{\sqrt{1+c_1^2}}C_\alpha\sigma, 0, \frac{1}{\sqrt{1+c_1^2}}C_\alpha\sigma)$, two dimensional views of the rejection region of (2.3). Left. The side of the curve ABCD, indicated by R, when $\gamma_1 = 0$. Right. The side of the curve EFCG, indicated by R, when $\gamma_2 = 0$.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_5$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - (\frac{b_1}{c_1}, 0, 0)\|^2 = (\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_2^2 \geq C_\alpha^2\sigma^2$, and hence the boundary of the rejection region is a partly cylindrical region with axis $\gamma_0 = \frac{b_1}{c_1}$, $\gamma_2 = 0$ and radius $C_\alpha\sigma$ contained in the acceptance region (the curve BC in Fig. 2 (left) is the two dimensional view on the $\gamma_0\gamma_2$-hyperplane).

The plane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ intersects the $\gamma_0\gamma_1$-plane (i.e. $\gamma_2 = 0$) at the line $c_1\gamma_0 + d_1\gamma_1 = b_1$ (line XL in Fig. 2 right). Perpendicular to this line at the point $\boldsymbol{L}$ on the $\gamma_0\gamma_1$-plane is the line $\gamma_1 = \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}$ (line LY in Fig. 2 (right)). The hyperplanes in $\mathbb{R}^3$ generated from the lines $c_1\gamma_0 + d_1\gamma_1 = b_1$ and $\gamma_1 = \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}$ are used to define the regions $\boldsymbol{S}_6, \boldsymbol{S}_7, \boldsymbol{S}_8$ below.

Consider when $\hat{\boldsymbol{\gamma}} \in \{\boldsymbol{\gamma} : \gamma_1 \geq 0, \gamma_2 < 0\} = \boldsymbol{S}_6 \uplus \boldsymbol{S}_7 \uplus \boldsymbol{S}_8$, where $\boldsymbol{S}_6 = \{0 \leq \gamma_1 < \frac{b_1}{d_1} - \frac{c_1}{d_1}\gamma_0, \gamma_2 < 0\}$, $\boldsymbol{S}_7 = \{\gamma_1 \geq \max\{\frac{b_1}{d_1} - \frac{c_1}{d_1}\gamma_0, \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}\}, \gamma_2 < 0\}$, and $\boldsymbol{S}_8 = \{0 \leq \gamma_1 < \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}, \gamma_2 < 0\}$, then $\boldsymbol{\gamma}^*$ is the projection of $\hat{\boldsymbol{\gamma}}$ onto the $\gamma_0\gamma_1$ hyperplane and $\bar{\boldsymbol{\gamma}}$, the projection of $\boldsymbol{\gamma}^*$ onto the edge of $\boldsymbol{G}_{01}$ on the hyperplane $\gamma_0\gamma_1$ (onto the line $c_1\gamma_0 + d_1\gamma_1 = b_1$ which is line XL in Fig. 2 (right)), if $\boldsymbol{\gamma}^* \notin \boldsymbol{G}_{01}$.

So when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_6$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\| = \|(\hat{\gamma}_0, \hat{\gamma}_1, 0) - (\hat{\gamma}_0, \hat{\gamma}_1, 0)\|^2 = 0$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_7$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|(\hat{\gamma}_0, \hat{\gamma}_1, 0) - ((\hat{\gamma}_0, \hat{\gamma}_1, 0) \cdot \boldsymbol{v})\boldsymbol{v}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{v}$ is a unit vector along the line $c_1\gamma_0 + d_1\gamma_1 = b_1$ on the $\gamma_0\gamma_1$-hyperplane, yields a hyperplane which is parallel and has $C_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + d_1\gamma_1 = b_1$. The equation of this new hyperplane is $c_1\gamma_0 + d_1\gamma_1 = b_1 + \sqrt{c_1^2 + d_1^2}C_\alpha\sigma$, the boundary of the rejection region in $\boldsymbol{S}_7$ (line EF in Fig. 2 (right) is the two dimensional view of the boundary of the rejection region when $\gamma_2 = 0$).

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_8$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|(\hat{\gamma}_0, \hat{\gamma}_1, 0) - (\frac{b_1}{c_1}, 0, 0)\|^2 = (\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1^2 \geq C_\alpha^2\sigma^2$, and hence the boundary of the rejection region in $\boldsymbol{S}_8$ is (the curve FC in Fig. 2 (right) is the two dimensional view on $\gamma_0\gamma_1$ plane) a cylindrical region with axis $\gamma_0 = \frac{b_1}{c_1}$, $\gamma_1 = 0$ and radius $C_\alpha\sigma$, contained in the acceptance region.

Next consider the equation of the hyperplane which is orthogonal to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ and contains the line $c_1\gamma_0 + \gamma_2 = b_1$ on the hyperplane $\gamma_1 = 0$, given by $c_1d_1\gamma_0 - (1 + c_1^2)\gamma_1 + d_1\gamma_2 = b_1d_1$.

Similarly consider the equation of the hyperplane which is orthogonal to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ and contains the line $c_1\gamma_0 + d_1\gamma_1 = b_1$ on the hyperplane $\gamma_2 = 0$, given by $c_1\gamma_0 + d_1\gamma_1 - (c_1^2 + d_1^2)\gamma_2 = b_1$.

The hyperplanes defined in last two paragraphs along with those used in the definitions of $\boldsymbol{S}_3 - \boldsymbol{S}_8$ are used to define $\boldsymbol{S}_9 - \boldsymbol{S}_{13}$ below. When $\hat{\boldsymbol{\gamma}} \in \{\boldsymbol{\gamma} : \gamma_1 \geq 0, \gamma_2 \geq 0\} = \boldsymbol{S}_9 \uplus \boldsymbol{S}_{10} \uplus \boldsymbol{S}_{11} \uplus \boldsymbol{S}_{12} \uplus \boldsymbol{S}_{13}$, where $\boldsymbol{S}_9 = \boldsymbol{G}_{01} = \{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \leq b_1, 0 \leq \gamma_1, 0 \leq \gamma_2\}$, $\boldsymbol{S}_{10} = \{0 \leq \gamma_1 \leq \frac{c_1d_1}{1+c_1^2}\gamma_0 + \frac{d_1}{1+c_1^2}\gamma_2 - \frac{b_1d_1}{1+c_1^2}, \gamma_2 \geq \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}$, $\boldsymbol{S}_{11} = \{\gamma_1 \geq \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}, 0 \leq \gamma_2 \leq \frac{c_1}{c_1^2+d_1^2}\gamma_0 + \frac{d_1}{c_1^2+d_1^2}\gamma_1 - \frac{b}{c_1^2+d_1^2}\}$, $\boldsymbol{S}_{13} = \{0 \leq \gamma_1 \leq \frac{d_1}{c_1}\gamma_0 - \frac{b_1d_1}{c_1^2}, 0 \leq \gamma_2 \leq \frac{1}{c_1}\gamma_0 - \frac{b_1}{c_1^2}\}$, and $\boldsymbol{S}_{12} = \{\gamma_1 \geq 0, \gamma_2 \geq 0\} - \boldsymbol{S}_9 \cup \boldsymbol{S}_{10} \cup \boldsymbol{S}_{11} \cup \boldsymbol{S}_{13}$ (see Fig. 3).

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_9$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\| = \|\hat{\boldsymbol{\gamma}} - \hat{\boldsymbol{\gamma}}\|^2 = 0$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{10}$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}} \cdot \boldsymbol{u})\boldsymbol{u}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{u}$ is a unit vector along the line $c_1\gamma_0 + \gamma_2 = b_1$, produces a curved plane which has $C_\alpha\sigma$ distance to the line $c_1\gamma_0 + \gamma_2 = b_1$, $\gamma_1 = 0$. Here the boundary of the rejection region is the surface of a cylinder whose axis is the line $c_1\gamma_0 + \gamma_2 = b$, $\gamma_1 = 0$ and the radius is $C_\alpha\sigma$. Let ω_2 be the acute angle between $c_1\gamma_0 + \gamma_2 = b_1$ and γ_0 axis on $\gamma_0\gamma_2$ hyperplane and then $\tan\omega_2 = c_1$. To get the equation of this cylinder, consider first the cylinder,

$$\gamma_1^2 + \left(\gamma_0 - \frac{b_1}{c_1}\right)^2 = C_\alpha^2\sigma^2, \tag{2.4}$$

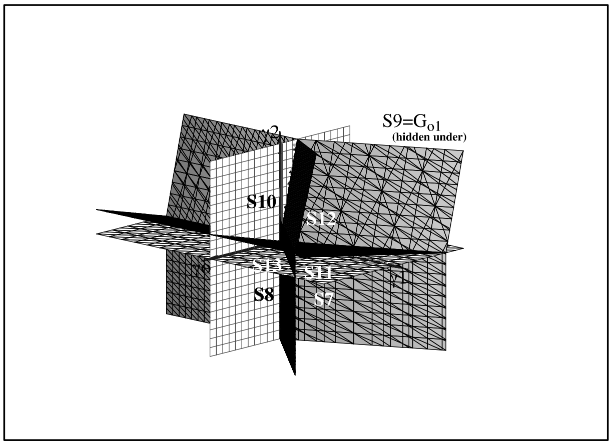

Fig. 3. Regions $\boldsymbol{S}_9$–$\boldsymbol{S}_{13}$ in $\{\boldsymbol{\gamma} : \gamma_1 \geq 0, \gamma_2 \geq 0\}$.

whose axis is the line $\gamma_0 = b_1/c_1$ on the $\gamma_0\gamma_2$ hyperplane and radius is $C_\alpha\sigma$. Now rotate this cylinder by an angle $\theta_1 = \frac{\pi}{2} - \omega_2$ counter clockwise about γ_1 axis. To get the equation of the rotated cylinder, first consider the transformation of the system of axis through the point $(\frac{b_1}{c_1}, 0, 0)$ by an angle θ_1 counter clockwise about the γ_1 axis. Then by using the corresponding rotation matrix we get,

$$\begin{bmatrix} \gamma_0 - \dfrac{b_1}{c_1} \\ \gamma_1 \\ \gamma_2 \end{bmatrix} \longrightarrow \begin{bmatrix} \cos\theta_1 & 0 & \sin\theta_1 \\ 0 & 1 & 0 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \gamma_0 - \dfrac{b_1}{c_1} \\ \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \left(\gamma_0 - \dfrac{b_1}{c_1}\right)\cos\theta_1 + \gamma_2 \sin\theta_1 \\ \gamma_1 \\ -\left(\gamma_0 - \dfrac{b_1}{c_1}\right)\sin\theta_1 + \gamma_2\cos\theta_1 \end{bmatrix}. \tag{2.5}$$

Then replacing $\gamma_0 - \frac{b_1}{c_1}, \gamma_1, \gamma_2$ in (2.4) by new coordinates, the equation of the rotated cylinder is $\gamma_1^2 + ((\gamma_0 - \frac{b_1}{c_1})\cos\theta_1 + \gamma_2 \sin\theta_1)^2 = C_\alpha^2\sigma^2$. Since $\omega_2 = \tan^{-1} c_1$, we have $\sin\theta_1 = \sin(\frac{\pi}{2} - \omega_2) = \frac{1}{\sqrt{1+c_1^2}}$ and $\cos\theta_1 = \cos(\frac{\pi}{2} - \omega_2) = \frac{c_1}{\sqrt{1+c_1^2}}$, so the equation is $\gamma_1^2 + (\frac{1}{\sqrt{1+c_1^2}}\gamma_2 + \frac{c_1}{\sqrt{1+c_1^2}}(\gamma_0 - \frac{b_1}{c_1}))^2 = C_\alpha^2\sigma^2$, the boundary of the rejection region in $\boldsymbol{S}_{10}$.

Similarly when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{11}$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}} \cdot \boldsymbol{v})\boldsymbol{v}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{v}$ is a unit vector along the line $c_1\gamma_0 + d_1\gamma_1 = b_1$. This produces a curved plane which has $C_\alpha\sigma$ distance to the line $c_1\gamma_0 + d_1\gamma_1 = b_1$, $\gamma_2 = 0$. Here the boundary of the rejection region is the surface of a cylinder whose axis is the line $c_1\gamma_1 + d_1\gamma_1 = b_1$, $\gamma_2 = 0$ and the radius is $C_\alpha\sigma$. Let ω_1 be the acute angle between $c_1\gamma_0 + d_1\gamma_1 = b_1$ and γ_0 axis on $\gamma_1 = 0$ hyperplane and then $\tan\omega_1 = \frac{c_1}{d_1}$. With similar technique as for $\boldsymbol{S}_{10}$, it can be seen that the boundary of the rejection region in $\boldsymbol{S}_{11}$ is $\gamma_2^2 + (\frac{d_1}{\sqrt{c_1^2+d_1^2}}\gamma_1 + \frac{c_1}{\sqrt{c_1^2+d_1^2}}(\gamma_0 - \frac{b_1}{c_1}))^2 = C_\alpha^2\sigma^2$.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{12}$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}} \cdot \boldsymbol{w})\boldsymbol{w}\|^2 \geq C_\alpha^2\sigma^2$, where $\boldsymbol{w}$ is a unit vector to the direction $\overrightarrow{LB}$, where LB is the projection of $\hat{\boldsymbol{\gamma}}$ onto the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ along the vector $(c_1, d_1, 1)$. This gives the hyperplane which is parallel and has $C_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1$ given by $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1 + \sqrt{1 + c_1^2 + d_1^2}C_\alpha\sigma$, the boundary of the rejection region in $\boldsymbol{S}_{12}$.

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{13}$, $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = (\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1^2 + \hat{\gamma}_2^2 \geq C_\alpha^2\sigma^2$, and hence the boundary of the rejection region in $\boldsymbol{S}_{13}$ is $(\gamma_0 - \frac{b_1}{c_1})^2 + \gamma_1^2 + \gamma_2^2 = C_\alpha^2\sigma^2$, which is part of a sphere with center at $\boldsymbol{L}$ and radius $C_\alpha\sigma$.

Note that C_α depends on ω_1, ω_2, so that $C_\alpha = C_\alpha(\omega_1, \omega_2)$. Combining the above regions appropriately, we get the rejection region as a combination of plane, cylindrical and spherical surfaces that envelop the $\boldsymbol{G}_{01}$ region. The above development is summarized as follows.

For testing $\boldsymbol{G}_0$ versus $\boldsymbol{G}_1 - \boldsymbol{G}_0$ using LRT (2.3), the acceptance region is convex and the rejection region is the union of following nine *disjoint* regions,

1. $\left\{\hat{\gamma}_0 \geq \dfrac{b_1}{c_1} + C_\alpha\sigma,\ \hat{\gamma}_1 < 0,\ \hat{\gamma}_2 < 0\right\}$,
2. $\left\{\left(\hat{\gamma}_0 - \dfrac{b_1}{c_1}\right)^2 + \hat{\gamma}_2^2 \geq C_\alpha^2\sigma^2,\ \hat{\gamma}_1 < 0,\ 0 \leq \hat{\gamma}_2 < \dfrac{1}{c_1}\hat{\gamma}_0 - \dfrac{b_1}{c_1^2}\right\}$,
3. $\left\{\left(\hat{\gamma}_0 - \dfrac{b_1}{c_1}\right)^2 + \hat{\gamma}_1^2 \geq C_\alpha^2\sigma^2,\ 0 \leq \hat{\gamma}_1 < \dfrac{d_1}{c_1}\hat{\gamma}_0 - \dfrac{b_1 d_1}{c_1^2},\ \hat{\gamma}_2 < 0\right\}$,

4. $\left\{\left(\hat{\gamma}_0 - \frac{b_1}{c_1}\right)^2 + \hat{\gamma}_1^2 + \hat{\gamma}_2^2 \geq C_\alpha^2\sigma^2,\ 0 \leq \hat{\gamma}_1 < \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2},\ 0 \leq \hat{\gamma}_2 < \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\right\}$,

5. $\left\{c_1\hat{\gamma}_0 + \hat{\gamma}_2 \geq b_1 + \sqrt{1+c_1^2}C_\alpha\sigma,\ \hat{\gamma}_1 < 0,\ \hat{\gamma}_2 \geq \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\right\}$,

6. $\left\{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 - b_1 \geq \sqrt{c_1^2+d_1^2}C_\alpha\sigma,\ \hat{\gamma}_1 \geq \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2},\ \hat{\gamma}_2 < 0\right\}$,

7. $\left\{\hat{\gamma}_1^2 + \left(\frac{1}{\sqrt{1+c_1^2}}\hat{\gamma}_2 + \frac{c_1}{\sqrt{1+c_1^2}}\left(\hat{\gamma}_0 - \frac{b_1}{c_1}\right)\right)^2 \geq C_\alpha^2\sigma^2,\ 0 \leq \hat{\gamma}_1 \leq \frac{c_1 d_1}{1+c_1^2}\hat{\gamma}_0 + \frac{d_1}{1+c_1^2}\hat{\gamma}_2 - \frac{b_1 d_1}{1+c_1^2},\right.$

$\left.\hat{\gamma}_2 \geq \frac{1}{c_1}\hat{\gamma}_0 - \frac{b_1}{c_1^2}\right\}$, (2.6)

8. $\left\{\hat{\gamma}_2^2 + \left(\frac{d_1}{\sqrt{c_1^2+d_1^2}}\hat{\gamma}_1 + \frac{c_1}{\sqrt{c_1^2+d_1^2}}\left(\hat{\gamma}_0 - \frac{b_1}{c_1}\right)\right)^2 \geq C_\alpha^2\sigma^2, \hat{\gamma}_1 \geq \frac{d_1}{c_1}\hat{\gamma}_0 - \frac{b_1 d_1}{c_1^2},\right.$

$\left.0 \leq \hat{\gamma}_2 \leq \frac{c_1}{c_1^2+d_1^2}\hat{\gamma}_0 + \frac{d_1}{c_1^2+d_1^2}\hat{\gamma}_1 - \frac{b_1}{c_1^2+d_1^2}\right\}$,

9. $\left\{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 - b_1 \geq C_\alpha\sigma\sqrt{1+c_1^2+d_1^2},\ \hat{\gamma}_1 \geq \max\left\{0,\ \frac{c_1 d_1}{1+c_1^2}\hat{\gamma}_0 + \frac{d_1}{1+c_1^2}\hat{\gamma}_2 - \frac{b_1 d_1}{1+c_1^2}\right\},\right.$

$\left.\hat{\gamma}_2 \geq \max\left\{0, \frac{c_1 d_1}{c_1^2+d_1^2}\hat{\gamma}_0 + \frac{d_1}{c_1^2+d_1^2}\hat{\gamma}_1 - \frac{b_1}{c_1^2+d_1^2}\right\}\right\}$,

where $C_\alpha = C_\alpha(\omega_1, \omega_2)$, ω_1 is the angle between $c_1\gamma_0 + \gamma_2 = b_1$ and $\gamma_2 = 0$ on the $\gamma_0\gamma_2$-plane, and ω_2 is the angle between $c_1\gamma_0 + d_1\gamma_1 = b_1$ and $\gamma_1 = 0$ on the $\gamma_0\gamma_1$-plane.

2.2. Least favorable distribution

To find the least favorable distribution of $\bar{\chi}_{01}^2$ in (2.3), write

$$\Pr(\text{LRT} \leq t) = \sum_{i=1}^{13} \Pr(\text{LRT} \leq t | \hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_i) \Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_i). \tag{2.7}$$

It is shown in Lemma 2L in Supplement, that the least favorable null value of $\bar{\chi}_{01}^2$ is attained at $\boldsymbol{\gamma} = \boldsymbol{L} = (b_1/c_1, 0, 0)$. When $\boldsymbol{\gamma} = \boldsymbol{L}$, $\hat{\boldsymbol{\gamma}} \sim \mathcal{N}_3(\boldsymbol{L}, \sigma^2\boldsymbol{I})$ and hence the length and the direction of the $\hat{\boldsymbol{\gamma}}$ are independent. Then for each region $\boldsymbol{S}_i$, $\Pr(\text{LRT} \leq t | \hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_i) = \Pr(\text{LRT} \leq t)$, $\forall t > 0$.

As shown earlier, LRT $= 0$ for $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_i$, $i = 1, 3, 6, 9$. When $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_2$, LRT $= (\hat{\gamma}_0 - \frac{b_1}{c_1})^2/\sigma^2$ is the squared length of the first coordinate and hence LRT has a χ_1^2 distribution. When $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_5, \boldsymbol{S}_8$, LRT is $((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_2^2)/\sigma^2$ and $((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1^2)/\sigma^2$ respectively, each of which is a summation of two squared lengths and hence each is distributed as a χ_2^2 distribution. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{13}$, LRT $= ((\hat{\gamma}_0 - \frac{b_1}{c_1})^2 + \hat{\gamma}_1^2 + \hat{\gamma}_2^2)/\sigma^2$, which is a summation of three squared lengths and hence is distributed as a χ_3^2 distribution.

When $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_4$, we consider a new orthogonal coordinate system, with axes along the lines $\gamma_0 - c_1\gamma_2 = \frac{b_1}{c_1}$ and $c_1\gamma_0 + \gamma_2 = b_1$ on $\gamma_1 = 0$ hyperplane as new γ_0 and γ_2 axes respectively, then the LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_4$ is the squared length of one of the coordinates only, and hence is distributed as a χ_1^2 distribution. Similarly when $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_7$, we consider the new orthogonal coordinate system, with axes along the lines $d_1\gamma_0 - c_1\gamma_1 = \frac{b_1 d_1}{c_1}$ and $c_1\gamma_0 + d_1\gamma_1 = b_1$, $\gamma_2 = 0$ hyperplane as new γ_0 and γ_1 axes respectively, then the LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_7$ is the squared length of one of the coordinates only and hence is distributed as a χ_1^2 distribution ($\boldsymbol{X} \sim \mathcal{N}_3(\boldsymbol{0}, \boldsymbol{I}_3)$ if and only if $\boldsymbol{PX} \sim \mathcal{N}_3(\boldsymbol{0}, \boldsymbol{I}_3)$ where $\boldsymbol{P}$ is a projection (rotation) matrix).

When $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_{10}$, $\boldsymbol{\gamma}^* = \hat{\boldsymbol{\gamma}}$ and $\bar{\boldsymbol{\gamma}}$ is the projection of $\boldsymbol{\gamma}^*$ onto the line $c_1\gamma_0 + \gamma_2 = b_1, \gamma_1 = 0$ (i.e., $\bar{\boldsymbol{\gamma}} = \Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01})$). So LRT $= \|\hat{\boldsymbol{\gamma}} - \Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01})\|^2/\sigma^2 = \|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01}^*)\|^2/\sigma^2$. Further the projection matrix $\boldsymbol{P}_1$ for the projection of $\hat{\boldsymbol{\gamma}}$ onto the line $c_1\gamma_0 + \gamma_2 = b_1, \gamma_1 = 0$ (on to the surface of $\boldsymbol{G}_{01}$) is $\boldsymbol{P}_1 = \boldsymbol{s}_1^\top\boldsymbol{s}_1$ [22], where $\boldsymbol{s}_1 = (-b_1, 0, b_1 c_1)$ is the directional vector of the line $c_1\gamma_0 + \gamma_2 = b_1, \gamma_1 = 0$. Here note that rank$(\boldsymbol{P}_1) = 1$. Further since here $\boldsymbol{G}_{01}$ and $\boldsymbol{G}_{01}^*$ are polyhedral cones, $\|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01}^*)\|^2/\sigma^2 \sim \chi_{3-1}^2$ ([20], p. 127) and hence the LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{10}$ has χ_2^2 distribution.

Similarly when $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_{11}$, $\boldsymbol{\gamma}^* = \hat{\boldsymbol{\gamma}}$ and $\bar{\boldsymbol{\gamma}}$ is the projection of $\boldsymbol{\gamma}^*$ onto the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$ (i.e., $\bar{\boldsymbol{\gamma}} = \Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01})$). So LRT $= \|\hat{\boldsymbol{\gamma}} - \Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01})\|^2/\sigma^2 = \|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}^*_{01})\|^2/\sigma^2$. Further the projection matrix $\boldsymbol{P}_2$ for the projection of $\hat{\boldsymbol{\gamma}}$ onto the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$ (onto the surface of $\boldsymbol{G}_{01}$) is $\boldsymbol{P}_2 = \boldsymbol{s}_2^\top \boldsymbol{s}_2$, where $\boldsymbol{s}_2 = (-b_1d_1, c_1d_1, 0)$ is the directional vector of the line $c_1\gamma_0 + d_1\gamma_1 = b_1, \gamma_2 = 0$. Here note that rank$(\boldsymbol{P}_2) = 1$ and then $\|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}^*_{01})\|^2/\sigma^2 \sim \chi^2_{3-1}$ [20] and hence the LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{11}$ has χ^2_2 distribution.

When $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_{12}$, LRT $= \|\hat{\boldsymbol{\gamma}} - \Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}_{01})\|^2/\sigma^2 = \|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}^*_{01})\|^2/\sigma^2$ which is the projection of $\hat{\boldsymbol{\gamma}}$ onto the line $(\frac{b_1}{c_1}, 0, 0) + u(c_1, d_1, 1)$, where u is a constant. Further the projection matrix $\boldsymbol{P}_3$ for the projection of $\hat{\boldsymbol{\gamma}}$ onto the line $(\frac{b_1}{c_1}, 0, 0) + u(c_1, d_1, 1)$ ($\boldsymbol{P}_3$ is the projection matrix for the projection onto the surface of $\boldsymbol{G}^*_{01}$) is $\boldsymbol{P}_3 = \boldsymbol{s}_3^\top \boldsymbol{s}_3$, where $\boldsymbol{s}_3 = (c_1, d_1, 1)$ is the directional vector of the line. Here note that rank$(\boldsymbol{P}_3) = 1$ and hence $\|\Pi(\hat{\boldsymbol{\gamma}}|\boldsymbol{G}^*_{01})\|^2/\sigma^2 \sim \chi^2_1$ [20]. Thus LRT given $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{12}$ has χ^2_1 distribution.

To find the probabilities $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_i)$, $1 \le i \le 12$, let $\boldsymbol{S}$ be a polyhedral cone in $\mathbb{R}^3$ which has three faces and vertex $\boldsymbol{L} = (b_1/c_1, 0, 0)$. Let $\theta_1, \theta_2, \theta_3$ be the angles between the faces of $\boldsymbol{S}$. Then $Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}) = (4\pi)^{-1}(\theta_1 + \theta_2 + \theta_3 - \pi)$. (Lemma 2 in Supplement.)

Here each $\boldsymbol{S}_i$, for $1 \le i \le 12$, is a polyhedral cone with three faces and vertex $\boldsymbol{L}$. Let us denote the angles between the faces of $\boldsymbol{S}_i$ as $\theta_{i,1}, \theta_{i,2}, \theta_{i,3}$.

Angles between the faces of $\boldsymbol{S}_1$ are $\theta_{1,1} = \pi/2$ (angle between the hyperplanes $\gamma_0 = b/c_1$ and $\gamma_1 = 0$), $\theta_{1,2} = \pi/2$ (angle between the hyperplanes $\gamma_0 = b/c_1$ and $\gamma_2 = 0$), $\theta_{1,3} = \pi/2$ (angle between the hyperplanes $\gamma_1 = 0$ and $\gamma_2 = 0$). Then, using Lemma 2, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_1) = (4\pi)^{-1}(\pi/2 + \pi/2 + \pi/2 - \pi) = 1/8$. By following similar argument, we can see that, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_2) = \Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_4) = \Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_7) = 1/8$. Similarly, it can be shown that $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_3) = (4\pi)^{-1}(\cos^{-1}\frac{1}{\sqrt{1+c_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_5) = (4\pi)^{-1}(\pi/2 - \cos^{-1}\frac{1}{\sqrt{1+c_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_6) = (4\pi)^{-1}(\cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_8) = (4\pi)^{-1}(\pi/2 - \cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_9) = (4\pi)^{-1}(\cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}} - \pi/2)$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{10}) = (4\pi)^{-1}(\cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{11}) = (4\pi)^{-1}(\cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}})$, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{12}) = (4\pi)^{-1}(\cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}})$, and finally, $\Pr(\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_{13})$ is found by subtraction from 1.

Collecting all these results and substituting in (2.7), we get the least favorable null value of $\bar{\chi}^2_{01}$ in (2.3) is attained at $\boldsymbol{\gamma} = \boldsymbol{L} = (b_1/c_1, 0, 0)$ and,

$$\sup_{\boldsymbol{\gamma} \in \boldsymbol{G}_{01}} \Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\| \ge C_\alpha \sigma\} = \Pr_{\boldsymbol{L}}\{\hat{\boldsymbol{\gamma}} : \|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\| \ge C_\alpha \sigma\}. \tag{2.8}$$

The least favorable null distribution of LRT is

$$\Pr(\text{LRT} \le t | \hat{\boldsymbol{\gamma}} = \boldsymbol{L}) = \sum_{i=0}^{3} w_i \Pr(\chi^2_i \le t), \tag{2.9}$$

where,

$$\begin{aligned}
w_0 &= (4\pi)^{-1}\left(\cos^{-1}\frac{1}{\sqrt{1+c_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{c_1^2+d_1^2}} + \cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}}\right), \\
w_1 &= (4\pi)^{-1}\left(\frac{3\pi}{2} - \cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}\right), \\
w_2 &= (4\pi)^{-1}\left(\pi + \cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} + \cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{1}{\sqrt{1+c_1^2}} - \cos^{-1}\frac{d_1 v}{\sqrt{c_1^2+d_1^2}}\right), \\
w_3 &= (4\pi)^{-1}\left(\frac{3\pi}{2} - \cos^{-1}\frac{\sqrt{1+c_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{\sqrt{c_1^2+d_1^2}}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{1}{\sqrt{1+c_1^2+d_1^2}}\right. \\
&\quad \left. - \cos^{-1}\frac{d_1}{\sqrt{1+c_1^2+d_1^2}} - \cos^{-1}\frac{d_1}{\sqrt{(1+c_1^2)(c_1^2+d_1^2)}}\right).
\end{aligned} \tag{2.10}$$

The result in (2.8) follows using standard techniques. It can be seen $w_0 + w_2 = w_1 + w_3 = \frac{1}{2}$ holds as well, with trigonometric calculations, as expected. The values of C_α derived from (2.9) are given in Table S1 (in Supplement).

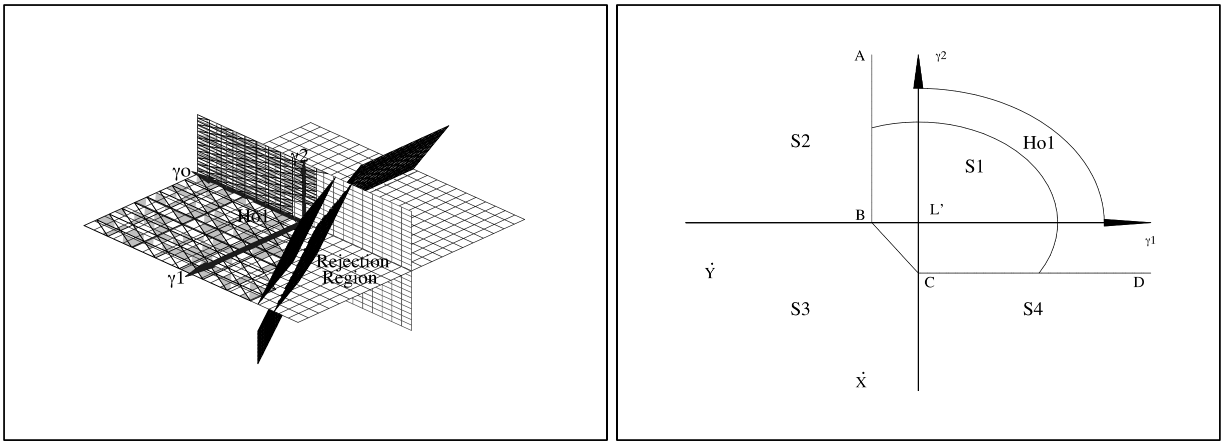

Fig. 4. *Left.* Region $\boldsymbol{H}_{01}$ and the rejection region of modified LRT of test (3.2). *Right.* Two dimensional view of the regions $\boldsymbol{S}_1$–$\boldsymbol{S}_4$ from the positive direction of γ_0 when $\gamma_0 = \frac{b'_1}{c_1}$.

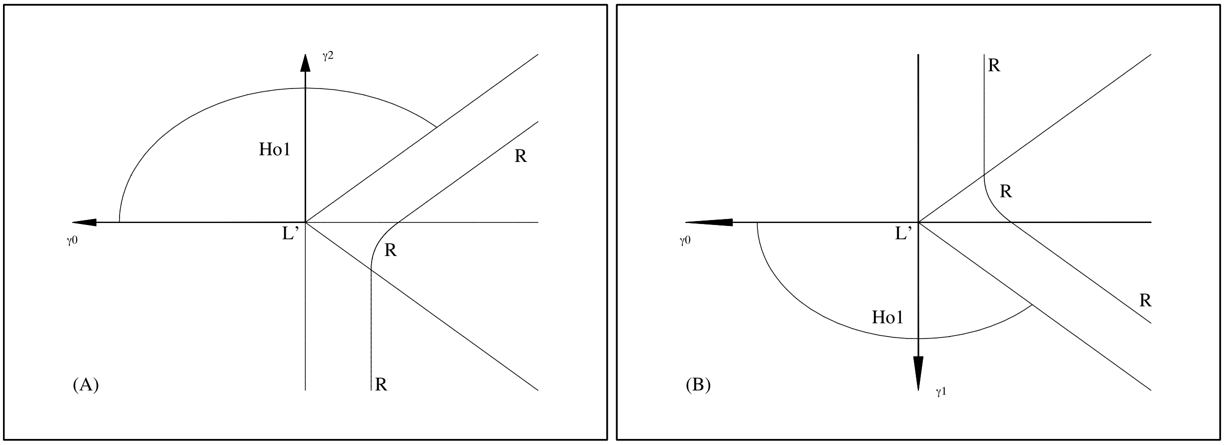

Fig. 5. Two-dimensional views of the rejection region of the LRT of (3.3), $\boldsymbol{L}' = (\frac{b'_1}{c_1}, 0, 0)$. Left: From the positive direction of γ_1 when $\gamma_1 = 0$. Right: From positive direction of γ_2 when $\gamma_2 = 0$.

3. Test in reverse direction of (2.1)

3.1. Rejection region when $x_{01} > 0, x_{02} > 0$

We now consider the hypotheses,

$$\boldsymbol{H}_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \geq u, \quad \beta_1 \geq 0,\ \beta_2 \geq 0, \qquad \boldsymbol{H}_1 : \beta_1 \geq 0,\ \beta_2 \geq 0, \tag{3.1}$$

for some $u \in \mathbb{R}$ and test $\boldsymbol{H}_0$ against $\boldsymbol{H}_1 - \boldsymbol{H}_0$. Define $b'_1 = \frac{S_{x_2}}{x_{02}} u \in \mathbb{R}$. The hypotheses in (3.1) can be written as,

$$\boldsymbol{H}_{01} : \gamma_2 \geq b'_1 - c_1\gamma_0 - d_1\gamma_1, \quad \gamma_1 \geq 0,\ \gamma_2 \geq 0, \qquad \boldsymbol{H}_1 : \gamma_1 \geq 0,\ \gamma_2 \geq 0. \tag{3.2}$$

The region $\boldsymbol{H}_{01}$ in $\boldsymbol{\gamma}$ space is now bounded by the faces $\{c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b'_1, \gamma_1 \geq 0, \gamma_2 \geq 0\}$, $\{\gamma_1 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \geq b'_1, \gamma_2 \geq 0\}$, and $\{\gamma_2 = 0, c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \geq b'_1, \gamma_1 \geq 0\}$ (see Fig. 4). The LRT rejects $\boldsymbol{H}_{01}$ for large values of the test statistic,

$$\bar{\chi}^2_{02} \equiv -2\log\Lambda = (\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2)/\sigma^2. \tag{3.3}$$

Let $\{\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 > D^2_\alpha\sigma^2\}$ be the rejection region for a level α test for some critical value D_α then Fig. 5 shows two dimensional views of the rejection region when $\gamma_1 = 0$ and $\gamma_2 = 0$ respectively (three dimensional picture looks messy here, hence omitted).

We show in the Supplement (Lemma 3) that the least favorable null value of LRT (3.3) is attained at $\lim_{t\to\infty, s\to\infty}(b'_1/c_1 - s - c_1 t, c_1 t, c_1 s)$ and,

$$\sup_{\boldsymbol{\gamma}\in\boldsymbol{H}_{01}} \Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 \geq D^2_\alpha\sigma^2\} = \lim_{t\to\infty, s\to\infty} \Pr_{(b'_1/c_1 - s - c_1 t, c_1 t, c_1 s)}\{\bar{\chi}^2_{02} > D^2_\alpha\}.$$

Further when $\sup_{\boldsymbol{\gamma}\in\boldsymbol{H}_{01}}$ is attained, the null critical value is $D^2_\alpha = \chi^2_{1,\alpha}$ (i.e. $D_\alpha = Z_\alpha$).

Next we consider a less restricted version of (3.2).

3.2. An unrestricted test

Ignoring the order restrictions $\gamma_1 \geq 0, \gamma_2 \geq 0$ in (3.2), consider the hypotheses

$$\boldsymbol{M}_{01} : c_1\gamma_0 + d_1\gamma_1 + \gamma_2 \geq b_1', \qquad \boldsymbol{M}_1 : c_1\gamma_0 + d_1\gamma_1 + \gamma_2 < b_1'. \tag{3.4}$$

The region $\boldsymbol{M}_{01}$ is now bounded by the face $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1'$ (see Fig S1 for region M_{01}, and the rejection region of (3.4), and Fig S2 for its two dimensional views in Supplement) and the LRT rejects $\boldsymbol{M}_{01}$ for small values of $\chi_{03} = \frac{c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 - b_1'}{\sqrt{1+c_1^2+d_1^2}\sigma}$. For level α, its rejection region is $\{\hat{\boldsymbol{\gamma}} : \chi_{03} < -Z_\alpha\sigma\}$, which is simplified as

$$c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 < b_1' - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma. \tag{3.5}$$

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{H}_1 - \boldsymbol{H}_{01}$, $\hat{\boldsymbol{\gamma}} = \boldsymbol{\gamma}^*$ and then the restricted test in (3.3) is $\bar{\chi}_{02}^2 = \|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2/\sigma^2$ simplifies as χ_{03}^2. The boundary of its rejection region is $\|\boldsymbol{\gamma}^* - \bar{\boldsymbol{\gamma}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}} \cdot \boldsymbol{w})\boldsymbol{w}\|^2 = D_\alpha^2\sigma^2$, where $\boldsymbol{w}$ is a unit vector to the direction $\overrightarrow{LB}$, where LB is the projection of $\hat{\boldsymbol{\gamma}}$ onto the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1'$. This gives a hyperplane which is parallel to and has $D_\alpha\sigma$ distance to the hyperplane $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1'$ and is given by

$$c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1' - \sqrt{1 + c_1^2 + d_1^2}D_\alpha\sigma, \tag{3.6}$$

which is the boundary of (3.5) since $D_\alpha = Z_\alpha$. Thus the restricted and unrestricted tests have the same boundary. But this is not the case when $\hat{\boldsymbol{\gamma}} \notin \boldsymbol{H}_1$.

In fact, the rejection region of the unrestricted LRT (3.4) contains that of the restricted LRT, which creates a philosophical dilemma when $\hat{\boldsymbol{\gamma}}$ is in some specific regions. To construct a modified rejection region which avoids the philosophical dilemma, we consider a partition of $\mathbb{R}^3$ as follows.

First note that the boundary of $\boldsymbol{H}_{01}$, $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1'$ meets hyperplane $\gamma_2 = 0$ on the line $c_1\gamma_0 + d_1\gamma_1 = b_1'$, $\gamma_2 = 0$ and hyperplane $\gamma_1 = 0$ on the line $c_1\gamma_0 + \gamma_2 = b_1'$, $\gamma_1 = 0$. Hyperplanes $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b' - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$ (line BC in Fig. 4 (right) when $\gamma_0 = b_1'/c_1$) and $c_1\gamma_0 + d_1\gamma_1 = b_1'$ ($\gamma_1 = 0$ when $\gamma_0 = b_1'/c_1$) intersect on the hyperplane $\hat{\gamma}_2 = -\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$ (line CD in Fig. 4 (right) is the two dimensional view when $\gamma_0 = \frac{b'}{c_1}$). Similarly, hyperplanes $c_1\gamma_0 + d_1\gamma_1 + \gamma_2 = b_1' - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$ and $c_1\gamma_0 + \gamma_2 = b_1'$ ($\gamma_2 = 0$ when $\gamma_0 = b_1'/c_1$) intersect on the hyperplane $\gamma_1 = -\frac{1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$ (line AB in Fig. 4 (right) is the two dimensional view when $\gamma_0 = \frac{b_1'}{c_1}$).

The equation of the hyperplane passing through points $B = (\frac{b_1'}{c_1}, -\frac{1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma, 0)$ and $C = (\frac{b_1'}{c_1}, 0, -\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma)$ and orthogonal to hyperplane $\gamma_0 = \frac{b_1'}{c_1}$ is $\gamma_2 = -d_1\gamma_1 - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$ (line BC in Fig. 4 (right) is the two dimensional view when $\gamma_0 = \frac{b_1'}{c_1}$).

Using these boundaries, we define $\boldsymbol{S}_1 = \mathbb{R}^3 - \boldsymbol{S}_2 \cup \boldsymbol{S}_3 \cup \boldsymbol{S}_4$, $\boldsymbol{S}_2 = \{\boldsymbol{\gamma} : \gamma_1 \leq -\frac{1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma, \gamma_2 \geq 0\}$, $\boldsymbol{S}_3 = \{\boldsymbol{\gamma} : \gamma_1 < 0, \gamma_2 < \min\{0, -d_1\gamma_1 - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma\}\}$, and $\boldsymbol{S}_4 = \{\boldsymbol{\gamma} : \gamma_1 \geq 0, \gamma_2 \leq -\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma\}$. Fig. 4 (right) shows a two dimensional view of above four disjoint regions when $\gamma_0 = b_1'/c_1$. Note that $\boldsymbol{S}_1$–$\boldsymbol{S}_4$ are different from those in Section 2.

Now note that when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_2 \cap \{b' - c_1\gamma_0 \leq \gamma_2 < b_1' - c_1\gamma_0 - d_1\gamma_1 - \sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma\}$, $\boldsymbol{H}_{01}$ is rejected even though $\boldsymbol{\gamma}^* \in \boldsymbol{H}_{01}$. Similar areas are present in $\boldsymbol{S}_3$ and in $\boldsymbol{S}_4$ which result in the same dilemma.

3.3. Construction of a modified test

We propose a modification of the rejection region of the restricted LRT as follows. We still use the boundary of the rejection region of the unrestricted test when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_1$. Since the hyperplane (3.6) and boundary of the region $\boldsymbol{S}_2$ ($\gamma_1 = -\frac{1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$) intersect on the plane $c_1\gamma_0 + \gamma_2 = b_1'$, $\gamma_1 = -\frac{1}{d_1}\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$, we propose $c_1\gamma_0 + \gamma_2 = b_1'$ as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_2$. Similarly, since hyperplane (3.6) and the boundary of the region $\boldsymbol{S}_4$ ($\gamma_2 = -\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$) intersect on the plane $c_1\gamma_0 + d_1\gamma_1 = b_1'$, $\gamma_2 = -\sqrt{1 + c_1^2 + d_1^2}Z_\alpha\sigma$, we propose $c_1\gamma_0 + d_1\gamma_1 = b_1'$ as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_4$. Since hyperplane (3.6) and the boundary of the region $\boldsymbol{S}_3$ ($\gamma_2 =$

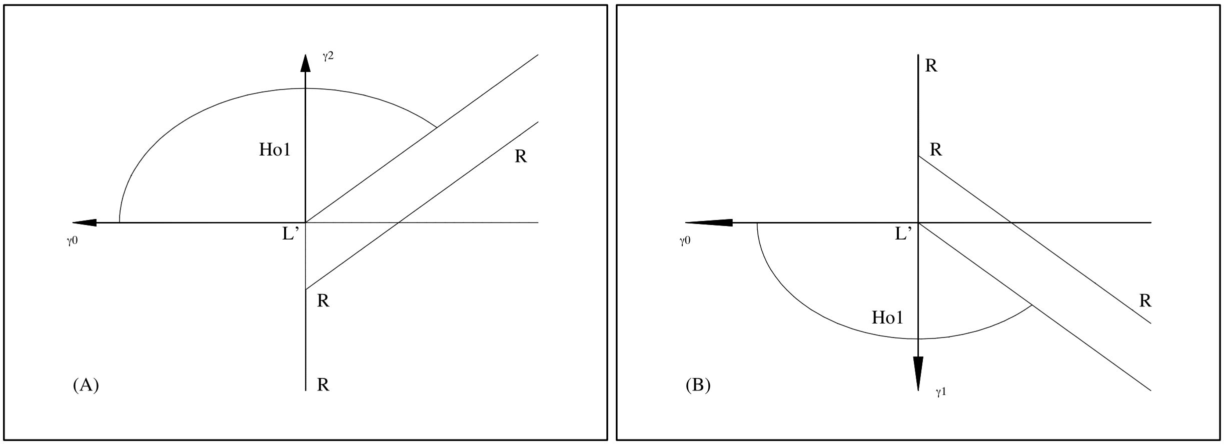

Fig. 6. Two 2-dimensional views of the modified rejection region. Left: From the positive direction of γ_1 when $\gamma_1 = 0$, and, Right: From positive direction of γ_2 when $\gamma_2 = 0$. $\boldsymbol{L}' = (\frac{b_1'}{c_1}, 0, 0)$. Compare with Fig. 5.

$-d_1\gamma_1 - \sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma, \gamma_0 = \frac{b_1'}{c_1})$ intersect on the line $\gamma_2 = -d_1\gamma_1 - \sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma, \gamma_0 = \frac{b_1'}{c_1}$, we propose $\gamma_0 = \frac{b_1'}{c_1}$ as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_3$. All the hyperplanes meet together in boundary.

So we reject $\boldsymbol{H}_{01}$ at level α when,

$$\begin{aligned}
&1.\ c_1\hat{\gamma}_0 + \hat{\gamma}_2 \le b_1' \quad \text{if } \hat{\gamma}_1 \le \frac{-1}{d_1}\sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma,\ \hat{\gamma}_2 \ge 0, \\
&2.\ c_1\hat{\gamma}_0 + d_1\hat{\gamma}_1 \le b_1' \quad \text{if } \hat{\gamma}_1 \ge 0,\ \hat{\gamma}_2 \le -\sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma, \\
&3.\ \hat{\gamma}_0 < \frac{b_1'}{c_1} \quad \text{if } \hat{\gamma}_1 \le 0,\ \hat{\gamma}_2 \le \min\{0, -d_1\hat{\gamma}_1 - \sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma\},\ \text{and,} \\
&4.\ \hat{\gamma}_0 + d_1\hat{\gamma}_1 + \hat{\gamma}_2 \le b_1' - \sqrt{1 + c_1^2 + d_1^2} Z_\alpha \sigma, \quad \text{otherwise.}
\end{aligned} \tag{3.7}$$

Fig. 4 (left) shows the boundary of the corresponding rejection region and Fig. 6 shows two 2-dimensional views of the boundary of the rejection region when $\gamma_1 = 0$ and $\gamma_2 = 0$ respectively. When $x_{01} < 0$ and $x_{02} < 0$, the boundaries of the rejection regions are obtained by inverting the boundary of the rejection regions of tests (2.2) and (3.2) (see [17]).

4. Rejection regions for mixed signs

The mixed sign case does not arise for one predictor.

4.1. Hypothesis (2.1) when $x_{01} > 0$ and $x_{02} < 0$

Let $l, u \in \mathbb{R}$, $b_2 = \frac{lS_{x_2}}{x_{02}} \in \mathbb{R}$, $c_2 = \frac{S_{x_2}}{x_{02}\sqrt{n}} < 0$ and $d_2 = \frac{x_{01}S_{x_2}}{x_{02}S_{x_1}} < 0$. Now the hypotheses in (2.1) in terms of $\boldsymbol{\gamma}$ are,

$$\boldsymbol{G}_{03} : \gamma_2 \ge b_2 - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \ge 0,\ \gamma_2 \ge 0 \quad \text{and} \quad \boldsymbol{G}_1 : \gamma_1 \ge 0, \gamma_2 \ge 0. \tag{4.1}$$

The region $\boldsymbol{G}_{03}$ is bounded by the faces $\{c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2, \gamma_1 \ge 0, \gamma_2 \ge 0\}$, $\{\gamma_1 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \le b_2, \gamma_2 \ge 0\}$, and $\{\gamma_2 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \ge b_2, \gamma_1 \ge 0\}$ (see Fig. 7). The LRT rejects $\boldsymbol{G}_{03}$ for large values of the test statistic,

$$\bar{\chi}_{04}^2 \equiv -2 \log \boldsymbol{\Lambda} = (\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2)/\sigma^2. \tag{4.2}$$

Let $\{\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 > E_\alpha^2\sigma^2\}$ be the rejection region for a level α test for some critical value E_α. When $\hat{\gamma}_2 \ge 0$, the rejection region $\{\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2/\sigma^2 \ge E_\alpha\}$ is a union of three *disjoint* regions, as in (4.6), except replacing critical value of E_α in place of F_α.

When $\hat{\gamma}_2 \ge 0$, the boundary of this rejection region is same as that in Fig. 7. When $\hat{\gamma}_2 < 0$, the rejection region of $\bar{\chi}_{04}^2$ does not yield simplified expressions. The three-dimensional picture is also messy here. Fig. 8 gives two two-dimensional views of $\boldsymbol{G}_{03}$ and the rejection region for this case. We show in the supplement (Lemma 4) that the null least favorable distribution of LRT in (4.2) is attained at $\lim_{\gamma_0 \to \infty}(\gamma_0, 0, b_2 - c_2\gamma_0)$, that is,

$$\sup_{\boldsymbol{\gamma} \in \boldsymbol{G}_{03}} \Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 \ge E_\alpha^2\sigma^2\} = \lim_{\gamma_0 \to \infty} \Pr_{(\gamma_0, 0, b_2 - c_2\gamma_0)}\{\bar{\chi}_{04}^2 > E_\alpha^2\sigma^2\}.$$

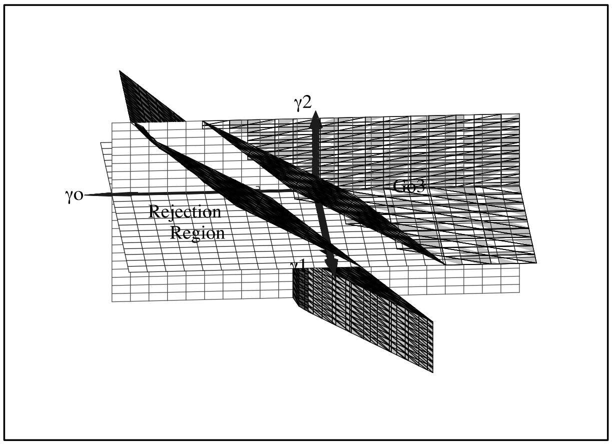

Fig. 7. Region $\boldsymbol{G}_{03}$ and the modified rejection region RR2 of test (4.1).

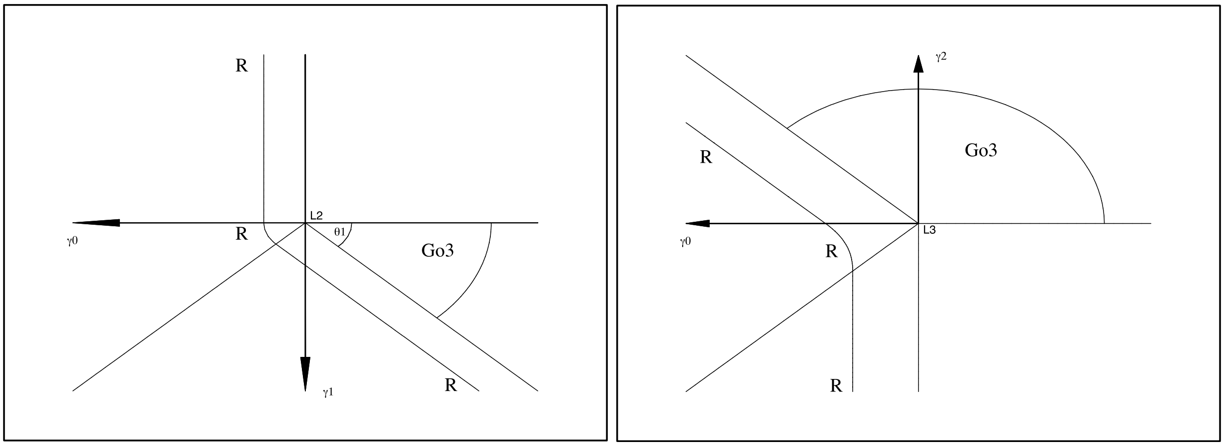

Fig. 8. Two dimensional views of the rejection region of LRT of (4.2) Left: On the hyperplane $\gamma_2 = \gamma_{2_0}$ from the positive direction of γ_2, and Right: From the positive direction of γ_1 when $\gamma_1 = 0$. Here $\boldsymbol{L}_3 = (b_2/c_2, 0, 0)$.

Further the null least favorable distribution of LRT is,

$$\sup_{\boldsymbol{\gamma} \in \boldsymbol{G}_{03}} \Pr(LRT \geq c) = \left(\frac{1}{4} + \frac{\theta_1}{2\pi}\right) P\left(\chi_0^2 \geq c\right) + \frac{1}{2} P(\chi_1^2 \geq c) + \left(\frac{1}{4} - \frac{\theta_1}{2\pi}\right) P(\chi_2^2 \geq c), \tag{4.3}$$

where θ_1 is the angle between the hyperplanes $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ and $\gamma_1 = 0$.

4.2. *Hypothesis (4.1) without the restriction $\gamma_2 \geq 0$*

Now consider the LRT for $\boldsymbol{M}_{02}$ versus $\boldsymbol{M}_{12} - \boldsymbol{M}_{02}$, where

$$\boldsymbol{M}_{02} : \gamma_2 \geq b_2 - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \geq 0 \quad \text{and} \quad \boldsymbol{M}_{12} : \gamma_1 \geq 0. \tag{4.4}$$

The region $\boldsymbol{M}_{02}$ is now bounded by the faces $\{c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \geq b_2, \gamma_1 = 0\}$ and $\{\gamma_1 \geq 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = 0\}$. The LRT rejects $\boldsymbol{M}_{02}$ for large values of

$$\bar{\chi}_{05}^2 = (\|\hat{\boldsymbol{\gamma}} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^{**}\|^2)/\sigma^2 = \|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2/\sigma^2, \tag{4.5}$$

where $\bar{\bar{\boldsymbol{\gamma}}}$ and $\boldsymbol{\gamma}^{**}$ are the MLEs under $\boldsymbol{M}_{02}$ and $\boldsymbol{M}_{12}$ respectively. Compare with (4.2).

Depending on the sign of $\hat{\gamma}_1$ and hyperplanes of interest, we divide $\mathbb{R}^3$ into five regions and calculate the test statistic $\bar{\chi}_{05}^2$ in (4.5) for each region separately. Finally we combine these regions to yield (4.6).

The hyperplanes $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ and $\gamma_1 = 0$ intersect on the line $c_2\gamma_0 + \gamma_2 = b_2, \gamma_1 = 0$. So we use hyperplane $c_2\gamma_0 + \gamma_2 = b_2$ in $\mathbb{R}^3$ to divide the region $\{\boldsymbol{\gamma} : \gamma_1 < 0\}$ into two *disjoint* regions (see Fig. 9, left) as $\boldsymbol{S}_1 \uplus \boldsymbol{S}_2$, where $\boldsymbol{S}_1 = \{\boldsymbol{\gamma} : \gamma_1 < 0, c_2\gamma_0 + \gamma_2 \geq b_2\}, \boldsymbol{S}_2 = \{\boldsymbol{\gamma} : \gamma_1 < 0, c_2\gamma_0 + \gamma_2 < b_2\}$. Call $c_2\gamma_0 + \gamma_2 = b_2, \gamma_1 = 0$ as the **center axis**; a line in $\mathbb{R}^3$ space (note that the regions $\boldsymbol{S}_i$'s are different from those in Sections 2 and 3).

Then from (4.5), when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_1$, $\|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - (\hat{\gamma}_0, 0, \hat{\gamma}_2)\|^2 = 0$. When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_2$, the rejection region is $\|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 = \|(\hat{\gamma}_0, 0, \hat{\gamma}_2) - ((\hat{\gamma}_0, 0, \hat{\gamma}_2)\cdot \boldsymbol{u})\boldsymbol{u}\|^2 \geq F_\alpha^2\sigma^2$, where $\boldsymbol{u}$ is a unit vector along the center axis, for some critical value F_α. This gives the plane which is parallel and has $F_\alpha\sigma$ distance to the hyperplane $c_2\gamma_0 + \gamma_2 = b_2$, the equation of the hyperplane is $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}F_\alpha\sigma$. So the boundary of the rejection region in $\boldsymbol{S}_2$ is $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}F_\alpha\sigma$ (line GC in Fig. 9 (right) is the two dimensional view when $\gamma_2 = 0$).

Next, the equation of the hyperplane which is orthogonal to $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ and contains the line $c_2\gamma_0 + \gamma_2 = b_2, \gamma_1 = 0$ (center axis) is $c_2d_2\gamma_0 - (1+c_2^2)\gamma_1 + d_2\gamma_2 = b_2d_2$. Using this hyperplane, and $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$, we divide the region $\{\boldsymbol{\gamma} : \gamma_1 \geq 0\}$ into three regions (see Fig. 9, left) as $\hat{\boldsymbol{\gamma}} \in \{\boldsymbol{\gamma} : \gamma_1 \geq 0\} = \boldsymbol{S}_3 \uplus \boldsymbol{S}_4 \uplus \boldsymbol{S}_5$ such that

$$\boldsymbol{S}_3 = \left\{\boldsymbol{\gamma} : 0 \leq \gamma_1 < \frac{c_2d_2}{1+c_2^2}\gamma_0 + \frac{d_2}{1+c_2^2}\gamma_2 - \frac{b_2d_2}{1+c_2^2}\right\},$$

$$\boldsymbol{S}_4 = \left\{\boldsymbol{\gamma} : \gamma_1 \geq \frac{c_2d_2}{1+c_2^2}\gamma_0 + \frac{d_2}{1+c_2^2}\gamma_2 - \frac{b_2d_2}{1+c_2^2}, \gamma_1 \geq \frac{b_2}{d_2} - \frac{c_2}{d_2}\gamma_0 - \frac{1}{d_2}\gamma_2\right\},$$

and

$$\boldsymbol{S}_5 = \left\{\boldsymbol{\gamma} : 0 \leq \gamma_1 < \frac{b_2}{d_2} - \frac{c_2}{d_2}\gamma_0 - \frac{1}{d_2}\gamma_2\right\}.$$

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_3$, $\|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}}\cdot\boldsymbol{u})\boldsymbol{u}\|^2 \geq F_\alpha^2\sigma^2$, $\boldsymbol{u}$ is a unit vector along the center axis. This produces the surface of a cylinder with center axis as its axis and radius $F_\alpha\sigma$. Then, using transformed coordinates (as done for $\boldsymbol{S}_{10}$ in (2.5)), the boundary of the rejection region in $\boldsymbol{S}_3$ is $\gamma_1^2 + (\frac{1}{\sqrt{1+c_2^2}}\gamma_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\gamma_0 - \frac{b_2}{c_2}))^2 = F_\alpha^2\sigma^2$ (curve CF in Fig. 9 (right) is the two dimensional view).

When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_4$, $\|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 = \|\hat{\boldsymbol{\gamma}} - (\hat{\boldsymbol{\gamma}}\cdot\boldsymbol{w})\boldsymbol{w}\|^2 \geq F_\alpha^2\sigma^2$, where $\boldsymbol{w}$ is a unit vector to the direction $\overrightarrow{LB}$, where LB is the projection of $\hat{\boldsymbol{\gamma}}$ onto the hyperplane $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2$ along the vector $(c, d, 1)$. This gives the hyperplane which is parallel and has $F_\alpha\sigma$ distance to hyperplane $c_2\gamma_1 + d_2\gamma_1 + \gamma_2 = b_2$. Then the boundary of the rejection region in $\boldsymbol{S}_4$ is $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2 - \sqrt{1+c_2^2+d_2^2}F_\alpha\sigma$ (line FE in Fig. 9 (right) is the two dimensional view). When $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_5$, $\|\boldsymbol{\gamma}^{**} - \bar{\bar{\boldsymbol{\gamma}}}\|^2 = \|\hat{\boldsymbol{\gamma}} - \hat{\boldsymbol{\gamma}}\|^2 = 0$.

Collecting all the above results, we get the boundary of the rejection region as a combination of three surfaces as stated below.

Let $\{\hat{\boldsymbol{\gamma}} : \|\bar{\bar{\boldsymbol{\gamma}}} - \boldsymbol{\gamma}^{**}\| > F_\alpha\sigma\}$ be the rejection region for a level α-test, for testing $\boldsymbol{M}_{02}$ vs $\boldsymbol{M}_{12} - \boldsymbol{M}_{02}$, for some critical value F_α, then the acceptance region is convex and the boundary of the rejection region is union of the following three disjoint regions (see Fig. 10 left),

$$\begin{aligned}
&1.\ c_2\hat{\gamma}_0 + \hat{\gamma}_2 \leq b_2 - \sqrt{1+c_2^2}F_\alpha\sigma, \quad \hat{\gamma}_1 < 0,\\
&2.\ \hat{\gamma}_1^2 + \left(\frac{1}{\sqrt{1+c_2^2}}\hat{\gamma}_2 + \frac{c_2}{\sqrt{1+c_2^2}}\left(\hat{\gamma}_0 - \frac{b_2}{c_2}\right)\right)^2 \geq F_\alpha^2\sigma^2, \quad 0 \leq \hat{\gamma}_1 \leq \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2},\\
&3.\ c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \leq b_2 - F_\alpha\sigma\sqrt{1+c_2^2+d_2^2}, \hat{\gamma}_1 \geq \max\left\{0, \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2}\right\}.
\end{aligned} \tag{4.6}$$

4.3. Construction of modified test of hypotheses (4.1)

Following a similar argument as of the first part of the proof of the Lemma 4 in Supplement, it follows that the least favorable distribution of the LRT for test (4.4) is attained on the center axis ($c_2\gamma_0 + \gamma_2 = b_2, \gamma_1 = 0$) and also the least favorable distribution of the LRT (4.5) is same as that of the LRT for test (4.1) (i.e., $E_\alpha = F_\alpha$). So boundaries of the rejection region for the LRT for test (4.1) and that of the LRT for test (4.4) are same in ($\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$) space and the rejection region for (4.4) contains the rejection region for (4.1) in ($\mathbb{R}\times\mathbb{R}\times\mathbb{R}^-$) space. So the LRT for test (4.4) is more powerful than the restricted LRT for the hypothesis (4.1). But this creates another philosophical dilemma when $\hat{\boldsymbol{\gamma}}$ is in some specific regions.

To identify those regions, note that the boundary of (4.1) and (4.4) is same when $\hat{\gamma}_1 < 0, \hat{\gamma}_2 > 0$, which is $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}E_\alpha\sigma$. Also note that when $\hat{\gamma}_2 < 0, \hat{\gamma}_0 < \frac{b_2}{c_2}$, then $\boldsymbol{\gamma}^* \in \boldsymbol{G}_{03}$ causing the philosophical dilemma. To avoid it, since $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}E_\alpha\sigma$ intersects with $\gamma_0 = \frac{b_2}{c_2}$ at $\gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$, we divide $\mathbb{R}^3$ into two regions $\boldsymbol{T}_1 = \{\hat{\boldsymbol{\gamma}} : \hat{\gamma}_2 \geq -\sqrt{1+c_2^2}E_\alpha\sigma\}$ and $\boldsymbol{T}_2 = \{\hat{\boldsymbol{\gamma}} : \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma\}$. Recall the regions $\boldsymbol{S}_1 - \boldsymbol{S}_5$ from Section 4.2. When using $\bar{\chi}_{04}^2$ in (4.2), note that for $\hat{\boldsymbol{\gamma}}$ in $\boldsymbol{S}_2 \cap \boldsymbol{T}_2$, $\boldsymbol{S}_3 \cap \boldsymbol{T}_2$ or $\boldsymbol{S}_4 \cap \boldsymbol{T}_2$, $\boldsymbol{G}_{03}$ is rejected even though $\boldsymbol{\gamma}^*$, restricted MLE under $\boldsymbol{G}_{13}$ is in $\boldsymbol{G}_{03}$.

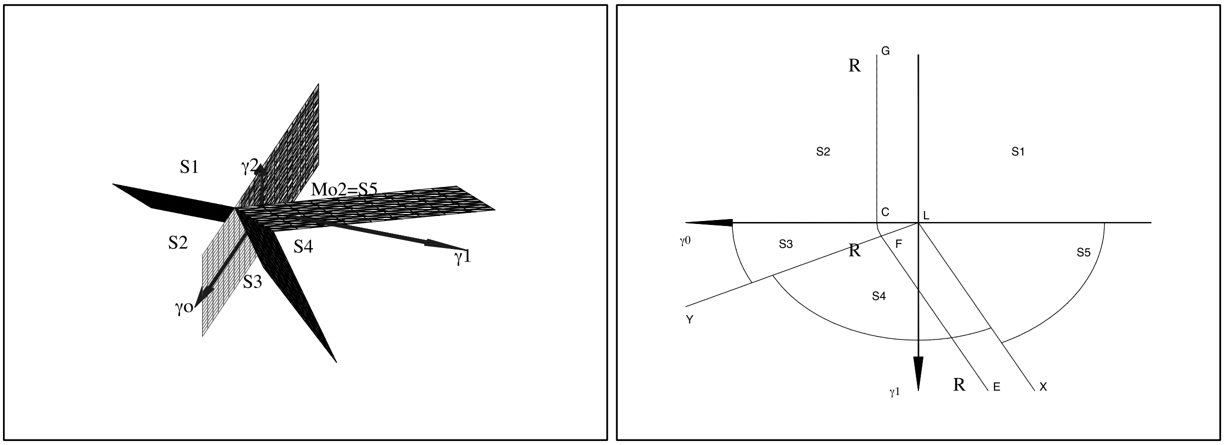

Fig. 9. Left. Regions $\boldsymbol{S}_1$–S_5. Two sides of $\boldsymbol{S}_4$ are perpendicular. Center axis is $c_2\gamma_0 + \gamma_2 = b_2, \gamma_1 = 0$. Right. Two dimensional view when $\gamma_2 = 0$ of the regions $\boldsymbol{S}_1$–S_5 and the rejection region of LRT of (4.4).

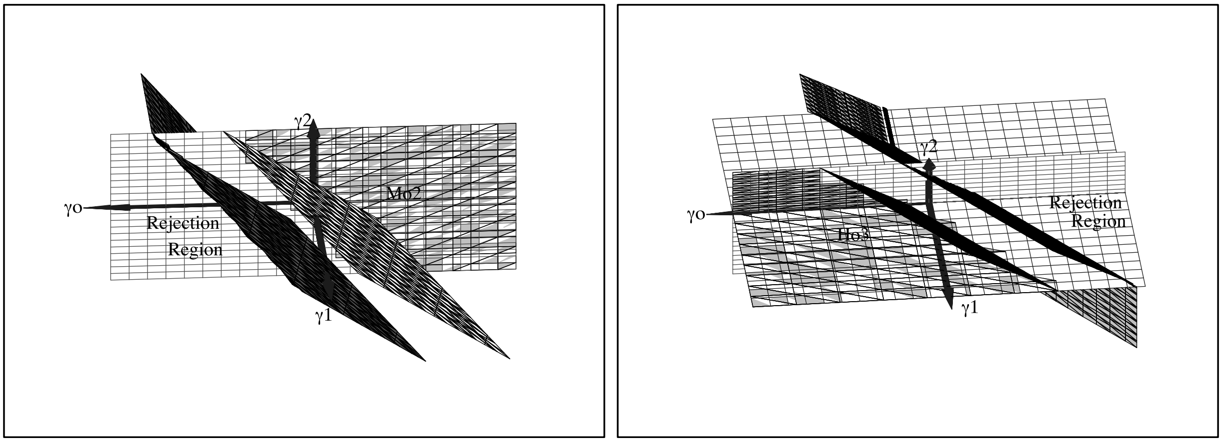

Fig. 10. Left. Region $\boldsymbol{M}_{02}$ and the rejection region of LRT of (4.4). Right. Region $\boldsymbol{H}_{03}$ and the rejection region of modified LRT of test (4.8).

So we propose a modification of the rejection region of the LRT of (4.1) as follows. When $\hat{\gamma}_2 \geq -\sqrt{1+c_2^2}E_\alpha\sigma$, we keep the same boundary of the LRT for test (4.4).

When $\hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma$, hyperplanes $c_2\gamma_0 + \gamma_2 = b_2 - \sqrt{1+c_2^2}E_\alpha\sigma$ and $\gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$ intersect on the line $(\frac{b_2}{c_2}, 0, -\sqrt{1+c_2^2}E_\alpha\sigma) + m(0, 1, 0)$, where m is a constant. So we propose the hyperplane $\gamma_0 = \frac{b_2}{c_2}$, as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in (\boldsymbol{S}_1 \cup \boldsymbol{S}_2) \cap \boldsymbol{T}_2$ (see Figs. 7 and 9 (left)).

Also the cylinder $\gamma_1^2 + (\frac{1}{\sqrt{1+c_2^2}}\gamma_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\gamma_0 - \frac{b_2}{c_2}))^2 = E_\alpha^2\sigma^2$ and the hyperplane $\gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$ intersect on the ellipse $\frac{\gamma_1^2}{(E_\alpha\sigma)^2} + \frac{(\gamma_0 - (\frac{b_2}{c_2} + \frac{\sqrt{1+c_2^2}}{c_2}E_\alpha\sigma))^2}{(\frac{\sqrt{1+c_2^2}}{c_2}E_\alpha\sigma)^2} = 1, \gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$. So we propose the curved plane (ellipsoid), which is parallel to γ_2 axis and contains the above ellipse, as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{S}_3 \cap \boldsymbol{T}_2$ (see Figs. 7 and 9 (left)). Further note that hyperplanes $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2 - \sqrt{1+c_2^2+d_2^2}F_\alpha\sigma$ and $\gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$ intersect on the line $c_2\gamma_0 + d_2\gamma_1 = b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_\alpha\sigma, \gamma_2 = -\sqrt{1+c_2^2}E_\alpha\sigma$. So we propose the hyperplane which is parallel to γ_2 axis and contains the above line, $c_2\gamma_0 + d_2\gamma_1 = b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_\alpha\sigma$ as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in (\boldsymbol{S}_4 \cup \boldsymbol{S}_5) \cap \boldsymbol{T}_2$ (see Figs. 7 and 9 (left)).

So we propose a modified test for testing $\boldsymbol{G}_{03}$ in (4.1), where we reject $\boldsymbol{G}_{03}$ at level α when

$$\begin{aligned}
&1.\ \hat{\gamma}_0 > \frac{b_2}{c_2} \quad \text{if } \hat{\gamma}_1 < 0, \text{ and } \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,\\
&2.\ c_2^2(\hat{\gamma}_0 - (b_2/c_2 + \sqrt{1+c_2^2}/c_2E_\alpha\sigma))^2 + (1+c_2^2)\hat{\gamma}_1^2 \ge (1+c_2^2)E_\alpha^2\sigma^2,\\
&\quad \text{if } 0 \le \hat{\gamma}_1 < \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2} \text{ and } \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,\\
&3.\ c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 \le b_2 - (\sqrt{1+c_2^2+d_2^2} - \sqrt{1+c_2^2})E_\alpha\sigma\\
&\quad \text{if } \hat{\gamma}_1 \ge \max\left\{0, \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2}\right\} \text{ and } \hat{\gamma}_2 < -\sqrt{1+c_2^2}E_\alpha\sigma,\\
&4.\ c_2\hat{\gamma}_0 + \hat{\gamma}_2 \le b_2 - \sqrt{1+c_2^2}E_\alpha\sigma, \quad \text{if } \hat{\gamma}_1 < 0 \text{ and } \hat{\gamma}_2 \ge -\sqrt{1+c_2^2}E_\alpha\sigma,\\
&5.\ \hat{\gamma}_1^2 + \left(\frac{1}{\sqrt{1+c_2^2}}\hat{\gamma}_2 + \frac{c_2}{\sqrt{1+c_2^2}}(\hat{\gamma}_0 - b_2/c_2)\right)^2 \ge E_\alpha^2\sigma^2,\\
&\quad \text{if } 0 < \hat{\gamma}_1 \le \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2} \text{ and } \hat{\gamma}_2 \ge -\sqrt{1+c_2^2}E_\alpha\sigma,\\
&6.\ c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \le b_2 - \sqrt{1+c_2^2+d_2^2}E_\alpha\sigma,\\
&\quad \text{if } \hat{\gamma}_1 \ge \max\left\{0, \frac{c_2d_2}{1+c_2^2}\hat{\gamma}_0 + \frac{d_2}{1+c_2^2}\hat{\gamma}_2 - \frac{b_2d_2}{1+c_2^2}\right\} \text{ and } \hat{\gamma}_2 \ge -\sqrt{1+c_2^2}E_\alpha\sigma.
\end{aligned} \tag{4.7}$$

4.4. *Hypothesis (3.1) when $x_{01} > 0$ and $x_{02} < 0$*

Let $b_2' = \frac{uS_{x_2}}{x_{02}} \in \mathbb{R}$. Now the test (3.1) in terms of $\boldsymbol{\gamma}$ is,

$$\boldsymbol{H}_{03} : 0 \le \gamma_2 < b_2' - c_2\gamma_0 - d_2\gamma_1, \quad \gamma_1 \ge 0, \quad \text{and} \quad \boldsymbol{H}_1 : \gamma_1 \ge 0, \gamma_2 \ge 0. \tag{4.8}$$

The region $\boldsymbol{H}_{03}$ is bounded by the faces $\{c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2', \gamma_1 \ge 0, \gamma_2 \ge 0\}$, $\{\gamma_1 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \ge b_2', \gamma_2 \ge 0\}$, and $\{\gamma_2 = 0, c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \le b_2', \gamma_1 \ge 0\}$. The LRT rejects $\boldsymbol{H}_{03}$ for large values of the test statistic,

$$\bar{\chi}_{06}^2 \equiv -2\log\boldsymbol{\Lambda} = (\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2)/\sigma^2. \tag{4.9}$$

Let $\{\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 > K_\alpha^2\sigma^2\}$ be the rejection region for a level α test for some critical value K_α. The least favorable null value of LRT in (4.9) is attained at $\lim_{\gamma_0 \to -\infty}(\gamma_0, \frac{b_2' - c_2\gamma_0}{d_2}, 0)$,

$$\sup_{\boldsymbol{\gamma} \in H_{03}} \Pr_{\boldsymbol{\gamma}}\{\hat{\boldsymbol{\gamma}} : \|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2 \ge K_\alpha^2\sigma^2\} = \lim_{\gamma_0 \to -\infty} \Pr_{\left(\gamma_0, \frac{b_2' - c_2\gamma_0}{d_2}, 0\right)}\{\bar{\chi}_{01}^2 > K_\alpha^2\sigma^2\}.$$

Further the least favorable distribution of LRT is,

$$\Pr(\text{LRT} \le c) = \left(\frac{1}{4} + \frac{\theta_2}{2\pi}\right)P(\chi_0^2 \le c) + \frac{1}{2}P(\chi_1^2 \le c) + \left(\frac{1}{4} - \frac{\theta_2}{2\pi}\right)P(\chi_2^2 \le c), \tag{4.10}$$

where θ_2 is the acute angle between two hyperplanes $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b_2'$ and $\gamma_2 = 0$.

4.5. *Ignoring the restriction $\gamma_1 \ge 0$ in (4.8)*

Ignoring the restriction $\gamma_1 \ge 0$ in (4.8), consider the test for $\boldsymbol{M}_{03}$ versus $\boldsymbol{M}_{13} - \boldsymbol{M}_{03}$, where

$$\boldsymbol{M}_{03} : 0 \le \gamma_2 < b_2' - c_2\gamma_0 - d_2\gamma_1, \qquad \boldsymbol{M}_{13} : \gamma_2 \ge 0. \tag{4.11}$$

The region $\boldsymbol{M}_{03}$ is now bounded by the faces $c_2\gamma_0 + \gamma_2 = b_2$ and $\gamma_2 = 0$ and LRT rejects $\boldsymbol{M}_{03}$ for large values of

$$\bar{\chi}_{07}^2 = (\|\hat{\boldsymbol{\gamma}} - \bar{\bar{\bar{\boldsymbol{\gamma}}}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^{***}\|^2)/\sigma^2 = \|\boldsymbol{\gamma}^{***} - \bar{\bar{\bar{\boldsymbol{\gamma}}}}\|^2/\sigma^2, \tag{4.12}$$

where $\bar{\bar{\bar{\boldsymbol{\gamma}}}}$, $\boldsymbol{\gamma}^{***}$ are the MLEs under $\boldsymbol{M}_{03}$ and $\boldsymbol{M}_{13}$ respectively. Fig. S3 in Supplement shows region $\boldsymbol{M}_{03}$, the rejection region of LRT of (4.11), and its two dimensional view.

Let $\{\hat{\boldsymbol{\gamma}} : \|\bar{\bar{\bar{\boldsymbol{\gamma}}}} - \boldsymbol{\gamma}^{***}\| > L_\alpha\sigma\}$ be the rejection region for a level α-test, for testing $\boldsymbol{M}_{03}$ vs $\boldsymbol{M}_{13} - \boldsymbol{M}_{03}$, for some critical value L_α, then the acceptance region is convex and the boundary of the rejection region is union of following three disjoint regions,

1. $c_2\gamma_0 + d_2\gamma_1 \le b'_2 + \sqrt{c_2^2 + d_2^2}L_\alpha\sigma$, $\gamma_2 < 0$,
2. $\gamma_2^2 + (\frac{d_2}{\sqrt{c_2^2+d_2^2}}\gamma_1 + \frac{c_2}{\sqrt{c_2^2+d_2^2}}(\gamma_0 - \frac{b'_2}{c_2}))^2 \ge L_\alpha^2\sigma^2$, $0 \le \gamma_2 \le \frac{c_2}{c_2^2+d_2^2}\gamma_0 + \frac{d_2}{c_2^2+d_2^2}\gamma_2 - \frac{b'_2}{c_2^2+d_2^2}$,
3. $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 \ge b'_2 + L_\alpha\sigma\sqrt{1 + c_2^2 + d_2^2}$, $\gamma_2 \ge \max\{0, \frac{c_2}{c_2^2+d_2^2}\gamma_0 + \frac{d_2}{c_2^2+d_2^2}\gamma_2 - \frac{b'_2}{c_2^2+d_2^2}\}$.

4.6. Construction of a modified test for (4.8)

The least favorable distribution of LRT for test (4.11) is attained on the line $c_2\gamma_0 + d_2\gamma_1 = b'_2$, $\gamma_2 = 0$. Also, the least favorable distribution of LRT for test (4.11) is same as that of LRT for test (4.8) (i.e., $K_\alpha = L_\alpha$). So boundaries of the rejection region for the LRT for test (4.8) and that of the LRT for test (4.11) are same in $(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$ space and rejection region for (4.8) contains the rejection region for (4.11) in $(\mathbb{R} \times \mathbb{R}^- \times \mathbb{R})$ space creating philosophical dilemma.

To identify those regions note that the hyperplanes $c_2\gamma_0+d_2\gamma_1 = b'_2+\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ (boundary of the rejection region of the unrestricted test (4.12) when $\hat{\gamma}_2 < 0$) and $\gamma_0 = \frac{b'_2}{c_2}$ intersect at $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$. Also when $\hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$, there is a $\hat{\boldsymbol{\gamma}}$ in rejection region with $\boldsymbol{\gamma}^*$ (MLE of $\hat{\boldsymbol{\gamma}}$ under $\boldsymbol{H}_1$) is in $\boldsymbol{H}_0$. So when $\hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ there are $\hat{\boldsymbol{\gamma}}$ which creates philosophical dilemma.

Consider the regions: $\boldsymbol{U}_1 = \{\boldsymbol{\gamma} : \gamma_2 < 0, c_2\gamma_0 + d_2\gamma_1 < b'_2\}$, $\boldsymbol{U}_2 = \{\boldsymbol{\gamma} : \gamma_2 < 0, c_2\gamma_0 + d_2\gamma_1 \ge b'_2\}$, $\boldsymbol{U}_3 = \{\boldsymbol{\gamma} : 0 \le \gamma_2 < \frac{c_2}{c_2^2+d_2^2}\gamma_0 + \frac{d_2}{c_2^2+d_2^2}\gamma_1 - \frac{b'_2}{c_2^2+d_2^2}\}$, $\boldsymbol{U}_4 = \{\boldsymbol{\gamma} : \gamma_2 \ge \frac{c_2}{c_2^2+d_2^2}\gamma_0 + \frac{d_2}{c_2^2+d_2^2}\gamma_1 - \frac{b_2}{c_2^2+d_2^2}, \gamma_2 \ge b'_2 - c_2\gamma_0 - d_2\gamma_1\}$, $\boldsymbol{U}_5 = \{\boldsymbol{\gamma} : 0 \le \gamma_2 < b'_2 - c_2\gamma_0 - d_2\gamma_1\}$.

Dividing $\mathbb{R}^3$ into $\boldsymbol{V}_1 = \{\mathbb{R}^3 : \gamma_1 \ge \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma\}$ and $\boldsymbol{V}_2 = \{\mathbb{R}^3 : \gamma_1 < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma\}$, we note that when $\hat{\boldsymbol{\gamma}}$ is in $\boldsymbol{U}_2 \cap \boldsymbol{V}_2$, $\boldsymbol{U}_3 \cap \boldsymbol{V}_2$ or $\boldsymbol{U}_4 \cap \boldsymbol{V}_2$, using $\bar{\chi}^2_{06}$, H_{03} is rejected even though $\boldsymbol{\gamma}^*$, restricted MLE under H_{13} is in H_{03}.

So we propose modification of LRT $\bar{\chi}^2_{06}$ as follows (see Fig. 10 right). We still use the boundary of the LRT for test (4.11) for the LRT of test (4.8) when $\hat{\gamma}_1 \ge \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$.

When $\hat{\gamma}_1 < \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$, note that hyperplanes $c_2\gamma_0+d_2\gamma_1 = b'_2+\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ and $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ intersect on the line $\gamma_0 = \frac{b'_2}{c_2}$, $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$. So we propose the hyperplane $\gamma_0 = \frac{b'_2}{c_2}$, which is parallel to γ_1 axis and contains the line $\gamma_0 = \frac{b'_2}{c_2}$, $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in (\boldsymbol{U}_1 \cup \boldsymbol{U}_2) \cap \boldsymbol{V}_2$.

Also the cylinder $\gamma_2^2 + (\frac{d_2}{\sqrt{c_2^2+d_2^2}}\gamma_1 + \frac{c_2}{\sqrt{c_2^2+d_2^2}}(\gamma_0 - \frac{b'_2}{c_2}))^2 = K_\alpha^2\sigma^2$ and the hyperplane $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ intersect on the ellipse $\frac{\gamma_2^2}{(K_\alpha\sigma)^2} + \frac{(\gamma_0 - (\frac{b'_2}{c_2} - \frac{\sqrt{c_2^2+d_2^2}}{c_2}K_\alpha\sigma))^2}{(\frac{\sqrt{c_2^2+d_2^2}}{c_2}K_\alpha\sigma)^2} = 1$, $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$. So we propose the curved plane (ellipsoid), which is parallel to γ_1 axis and contains the above ellipse as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in \boldsymbol{U}_3 \cap \boldsymbol{V}_2$. Further note that hyperplanes $c_2\gamma_0 + d_2\gamma_1 + \gamma_2 = b'_2 + \sqrt{1 + c_2^2 + d_2^2}K_\alpha\sigma$ and $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$ intersect on the line $c_2\gamma_0 + \gamma_2 = b'_2 + (\sqrt{1 + c_2^2 + d_2^2} - \sqrt{c_2^2 + d_2^2})K_\alpha\sigma$, $\gamma_1 = \frac{1}{d_2}\sqrt{c_2^2 + d_2^2}K_\alpha\sigma$. So we propose the hyperplane which is parallel to the γ_1 axis and contains the above line as the boundary of the rejection region when $\hat{\boldsymbol{\gamma}} \in (\boldsymbol{U}_4 \cup \boldsymbol{U}_5) \cap \boldsymbol{V}_2$ (see Fig. 14, supplement). So we reject H_{03} at level α when,

1. $\hat{\gamma}_0 \le \dfrac{b'_2}{c_2}$ if $\hat{\gamma}_1 < \dfrac{1}{d_2}\sqrt{c_2^2 + d_2^2}$, and $\hat{\gamma}_2 < 0$,
2. $c_2^2(\hat{\gamma}_0 - (b'_2/c_2 - \sqrt{c_2^2 + d_2^2}/c_2K_\alpha\sigma))^2 + (c_2^2 + d_2^2)\hat{\gamma}_2^2 \ge (c_2^2 + d_2^2)K_\alpha^2\sigma^2$,
   if $\hat{\gamma}_1 < \dfrac{1}{d_2}\sqrt{c_2^2 + d_2^2}$ and $0 \le \hat{\gamma}_2 < \dfrac{c_2}{c_2^2 + d_2^2}\hat{\gamma}_0 + \dfrac{d_2}{c_2^2 + d_2^2}\hat{\gamma}_2 - \dfrac{b'_2}{c_2^2 + d_2^2}$,
3. $c_2\gamma_0 + \gamma_2 \ge b'_2 + (\sqrt{1 + c_2^2 + d_2^2} - \sqrt{c_2^2 + d_2^2})K_\alpha\sigma$,
   if $\hat{\gamma}_1 < \dfrac{1}{d_2}\sqrt{c_2^2 + d_2^2}$ and $\hat{\gamma}_2 \ge \dfrac{c_2}{c_2^2 + d_2^2}\hat{\gamma}_0 + \dfrac{d_2}{c_2^2 + d_2^2}\hat{\gamma}_2 - \dfrac{b'_2}{c_2^2 + d_2^2}$,

$$4.\ c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 \le b_2' \quad \text{if } \hat{\gamma}_1 \ge \frac{1}{d_2}\sqrt{c_2^2+d_2^2},\ \text{and } \hat{\gamma}_2 < 0, \tag{4.13}$$

$$5.\ \hat{\gamma}_2^2 + \left(\frac{d_2}{\sqrt{c_2^2+d_2^2}}\hat{\gamma}_1 + \frac{c_2}{\sqrt{c_2^2+d_2^2}}(\hat{\gamma}_0 - b_2'/c_2)\right)^2 \ge K_\alpha^2\sigma^2$$

$$\text{if } \hat{\gamma}_1 \ge \frac{1}{d_2}\sqrt{c_2^2+d_2^2} \text{ and } 0 \le \hat{\gamma}_2 < \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_2 - \frac{b_2'}{c_2^2+d_2^2},$$

$$6.\ c_2\hat{\gamma}_0 + d_2\hat{\gamma}_1 + \hat{\gamma}_2 \ge b_2 + \sqrt{1+c_2^2+d_2^2}K_\alpha\sigma,$$

$$\text{if } \hat{\gamma}_1 \ge \frac{1}{d_2}\sqrt{c_2^2+d_2^2},\ \text{and } \hat{\gamma}_2 \ge \frac{c_2}{c_2^2+d_2^2}\hat{\gamma}_0 + \frac{d_2}{c_2^2+d_2^2}\hat{\gamma}_2 - \frac{b_2'}{c_2^2+d_2^2}.$$

The situation of $x_{01} < 0, x_{02} > 0$ can be derived by mirror images of $x_{01} > 0, x_{02} < 0$.

5. Confidence interval for $\beta_0 + \beta_1 x_{01} + \beta_2 x_{02}$

The rejection region for all cases can be expressed in original variables (see [17]). We define a $(1-\alpha)100\%$ confidence interval $[L, U]$ for the regression function at a given point (x_{01}, x_{02}) by inverting the tests (2.2) and (3.1): $L = \min\{l|\boldsymbol{G}_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \le l, \beta_1 \ge 0, \beta_2 \ge 0$ is accepted at level $\alpha/2$ against $\boldsymbol{G}_1 - \boldsymbol{G}_0\}$, $U = \max\{u|\boldsymbol{H}_0 : \beta_0 + \beta_1 x_{01} + \beta_2 x_{02} \ge u, \beta_1 \ge 0, \beta_2 \ge 0$ is accepted at level $\alpha/2$ against $\boldsymbol{H}_1 - \boldsymbol{H}_0\}$.

The expressions for L, U are evaluated for all cases of x_{01}, x_{02}, see supplemental material for these expressions (details in [17]). Although these expressions are long, nice symmetries appear in them. For example, when $x_{01} > 0$, $x_{02} > 0$, it is found that the *lower* boundary of the confidence interval for $E(Y)$ is of the general form $\hat{\beta}_0 + a_1\hat{\beta}_1 x_{01} + a_2\hat{\beta}_2 x_{02} - g(C_{\alpha/2}, S_{x_1} S_{x_2}, x_{01}, x_{02}, \hat{\beta}_1, \hat{\beta}_2)$, for some function g, if $\hat{\beta}_1 \in \boldsymbol{S}_1$ and $\hat{\beta}_2 \in \boldsymbol{S}_2$, where a_i is either 0 or 1, $i = 1, 2$ and $\boldsymbol{S}_j \subset \mathbb{R}^2$, $j = 1, 2$. Then for $x_{01} < 0$, $x_{02} < 0$, the *upper* boundary of the confidence interval for $E(Y)$ is $\hat{\beta}_0 + a_1\hat{\beta}_1 x_{01} + a_2\hat{\beta}_2 x_{02} + g(C_{\alpha/2}, S_{x_1} S_{x_2}, x_{01}, x_{02}, \hat{\beta}_1, \hat{\beta}_2)$ if $\hat{\beta}_1 \in \boldsymbol{S}_1$ and $\hat{\beta}_2 \in \boldsymbol{S}_2$. Such symmetries are repeated for all sign combinations of x_{01}, x_{02}.

6. σ^2 unknown

6.1. *When* $x_{01} > 0, x_{02} > 0$

Considering the hypotheses (2.1), the LRT is $\boldsymbol{\Lambda} = (\sigma^{*2}/\bar{\sigma}^2)^{n/2}$, where σ^{*2} and $\bar{\sigma}^2$ are the MLEs of σ^2 under $\boldsymbol{G}_1$ and $\boldsymbol{G}_0$, respectively. The LRT rejects $\boldsymbol{G}_0$ for large values of, $\lambda = 1 - \Lambda^{2/n} = 1 - \frac{\nu S^2 + \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2}{\nu S^2 + \|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2} = \frac{\|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2}{\nu S^2 + \|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2}$. Consider the test statistic $\boldsymbol{S}_{01} = \frac{\nu\boldsymbol{\Lambda}}{1-\boldsymbol{\Lambda}} = \frac{\|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2}{S^2 + \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2/\nu}$, a strictly increasing function of $\boldsymbol{\Lambda}$. [16] has discussed that the distribution of $\boldsymbol{S}_{01}$ is very difficult to determine (for least favorable or otherwise) in the one-predictor case. Following their suggestion, we propose the test: reject $\boldsymbol{G}_0$ if $T_1^2 \equiv \frac{\|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2}{S^2}$ is large, for our forgoing discussion. This test reduces to the $\bar{\chi}_{01}^2$ test in (2.3) when σ^2 is known. It may also be viewed as an approximation of the LRT for large ν.

As in σ known case, the least favorable distribution attains at $\boldsymbol{\gamma} = \boldsymbol{L}$. Using this value of $\boldsymbol{\gamma}$, $[(\|\hat{\boldsymbol{\gamma}} - \boldsymbol{L}\|^2/\sigma^2)/3]/(S^2/\sigma^2) \sim F_{3,n-3}$, where $F_{k,m}$ is an F random variable with k, m degrees of freedom. Using the same argument as in the σ known case it follows that the least favorable null distribution of T_1^2 is

$$P(T_1^2 \le C_\alpha|\boldsymbol{\gamma} = \boldsymbol{L}) = w_0 + \sum_{i=1}^{3} w_i P(F_{i,n-3} \le C_\alpha/i),$$

where w_i's are the weights in (2.10), and the critical value C_α solves

$$\alpha = w_1 P(F_{1,n-3} > C_\alpha^2) + w_2 P(F_{2,n-3} > C_\alpha^2/2) + w_3 P(F_{3,n-3} > C_\alpha^2/3) \tag{6.1}$$

for given α. Note that here C_α also depends on ν, that is, $C_\alpha = C_\alpha(\omega_1, \omega_2, \nu)$. Also note that C_α in the σ known case is $C_\alpha(\omega_1, \omega_2, \infty)$, hence the use of the same notation. The solution C_α of (6.1), as a function of ω_1, ω_2 and ν, is tabulated in Table S2 in supplement.

If the correct rejection region is given by $\{\hat{\boldsymbol{\gamma}} : \|\bar{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\| > C_\alpha S\}$, then the acceptance region is convex as it was in the σ known case and the boundary of the rejection region can be obtained from the σ known case replacing σ with S for each case and using the C_α values from Table S2 according to the value of ν.

We now consider our second main hypothesis (3.1). The LRT rejects $\boldsymbol{H}_0$ for large values of the test statistic, $S_{01}' = \frac{\|\hat{\boldsymbol{\gamma}} - \bar{\boldsymbol{\gamma}}\|^2 - \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2}{S^2 + \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}^*\|^2/\nu}$. Replacing σ with S, arguments similar to the σ known case show that when $\sup_{\boldsymbol{\gamma} \in \boldsymbol{H}_{01}}$ is attained, $D_\alpha = t_{\nu,\alpha}$. We propose the modification of the LRT as: to reject $\boldsymbol{H}_0$ at level α when (3.7) holds (replacing σ with S and Z_α with $t_{\nu,\alpha}$). When $x_{01} < 0, x_{02} < 0$, rejection regions of hypotheses (2.1), (3.1) are mirror images of those of (3.1), (2.1), respectively.

6.2. When $x_{01} > 0$ and $x_{02} < 0$

Results in Section 4.1 hold except replace χ_k^2 by $F_{k,n-3}$. Further here note that $E_\alpha = E_\alpha(\theta_1, \nu) = C_\alpha(\theta_1, \pi/2, \nu)$ (i.e., E_α is a special case of C_α with $\omega_1 = \theta_1$ and $\omega_2 = \pi/2$). So values for $E_\alpha(\theta_1, \nu)$ can also be obtained from Table S2. A modified test is proposed in this case [17] similar to the σ known case with proper replacements. For hypotheses (4.8), $K_\alpha = K_\alpha(\theta_2, \nu) = C_\alpha(\pi/2, \theta_2, \nu)$ (i.e. K_α is also a special case of C_α with $\omega_1 = \pi/2$ and $\omega_2 = \theta_2$). So values for $K_\alpha(\theta_2, \nu)$ can also be obtained from Table S2. A modified test is proposed in this case also as above [17].

7. Example

In Section 1.1, we introduced the SENIC data, which we analyze below. The variables of interest are the length of patient stay in a hospital in days (Y) as a function of patient's age in years (X_1') and infection risk (in percent) (X_2') given by [11]. Here we suspect β_1 is positive and β_2 is negative. Our goal is to construct confidence interval for E(Y) for given values of (x_{01}, x_{02}).

In this data with 113 observations, values of (X_1') range from 38.8 to 65.9, and, values of (X_2') range from 1.3 to 7.8. We form the matrix $X_{(113\times 3)}$ with the ith row given by $(1, X_1, X_2)$ where $X_{i,j+1} = X_{i,j}' - \bar{X}_{i,j}' = X_{i,j}' - (53.23,\ 4.35)$ for $i = 1, \ldots, 113,\ j = 1, 2$, so that $\sum X_{i2} = 0, \sum X_{i3} = 0$, for $i = 1, \ldots, 113$. Further, corr$(X_2, X_3) = 0.0010$, and $\sum X_{i2}X_{i3} = .73$ which is close to 0, so that X_2, X_3 are almost orthogonal.

Ordinary least squares regression estimators for the regression model Y with X_1 and X_2 (second and third columns of $X_{(113\times 3)}$, respectively) are $\hat{\beta}_0 = 57.00$, $\hat{\beta}_1 = 0.28$ and $\hat{\beta}_2 = -5.16$ and also $S = 32.29$, $S_{x_1}^2 = 2229.47$ and $S_{x_2}^2 = 201.38$.

To show the calculations for the $x_{01} > 0, x_{02} < 0$ case, let $(x_{01}, x_{01})^\top = (2, 0.5)$ or $(x_{01}', x_{02}')^\top = (x_{01} + 53.2319, x_{02} + 4.3549)^\top = (55.2319, 4.8549)^\top$. Then $c_2 = \frac{S_{x_2}}{x_{02}\sqrt{n}} = 2.6699$, $d_2 = \frac{S_{x_2}x_{01}}{x_{02}S_{x_1}} = 1.2022$, $\theta_1 = \cos^{-1}(\frac{d_2}{\sqrt{1+c^2+d^2}}) = 1.1718$, $\theta_2 = \cos^{-1}(\frac{1}{\sqrt{1+c_2^2+d_2^2}}) = 1.2417$. Setting $\theta_k = i_k\frac{\pi}{12}, k = 1, 2$ we find $i_1 = 4.4758$ and $i_2 = 4.7429$. Then we find $E_{0.025}(4.48\pi/12,\ 111) = 2.1287$ and $K_{0.025}(4.74\pi/12,\ 111) = 2.1074$ from Table $S2$ (with $i = 6$) by linear interpolation. Similar calculations for other sign choices of x_{01}, x_{02} yield table below.

The following table gives the 95% confidence intervals for $E(Y|x_{01}, x_{02})$, for different choices of x_{01}, x_{02}.

(x_{01}, x_{02})	Two predictors		First predictor		Second predictor	
	Restricted CI	Unrestricted CI	Restricted CI	Unrestricted CI	Restricted CI	Unrestricted CI
(2, 0.5)	(50.14, 62.39)	(47.99, 61.94)	(50.27, 64.27)	(50.82, 64.27)	(48.01, 61.24)	(48.02, 60.82)
(−2, −0.5)	(51.61, 63.86)	(52.06, 66.01)	(49.73, 63.73)	(49.73, 63.18)	(52.76, 65.99)	(53.18, 65.99)
(1.5, −0.3)	(52.16, 65.46)	(52.47, 65.46)	(50.56, 63.88)	(50.94, 63.88)	(52.15, 64.70)	(52.40, 64.70)
(−1.5, 0.3)	(48.54, 61.84)	(48.54, 61.53)	(50.12, 63.59)	(50.12, 63.06)	(49.30, 61.85)	(49.30, 61.60)

We find that the lengths of these confidence intervals depend on the values of (x_{01}, x_{02}) chosen.

8. Discussion

Statistical inference for the mean of the regression function is considered when the coefficients are nonnegative (or nonpositive) in the presence of two orthogonal predictors. The solutions are found using simple tools from calculus and geometry. Although formulas derived are longer than the corresponding ones in the unrestricted (or restricted one predictor) case, symmetry between different hypotheses regions helps substantially to deduce the formulas. Exact formulas of the level probabilities under the least favorable distributions are found. Further research needs to be done regarding correlated predictors, prediction intervals, lack-of-fit tests, and diagnostics for two predictor case. We hope our research will inspire other work in these important areas of regression analysis under monotonicity of predictors.

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Appendix A. Supplementary data

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.jmva.2016.07.008.

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Further reading

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