

# High School Mathematics Field Day Exam

Tuesday, February 16, 2010

Southern Illinois University Carbondale

## ***PLEASE READ THESE DIRECTIONS CAREFULLY!***

1. Calculators, cell phones, iPods or any other electronic devices **ARE NOT** permitted. **Any calculator, cell phone, iPod or any other electronic device seen at any time after entering the auditorium will immediately disqualify that student.**
2. Use a pencil with No. 2 lead for marking the answer sheet. Only marks in the small circles on the answer sheet are recorded. Avoid smudges. Keep all pencil marks inside the little circles. Do not try to cover up the letter. If the letter is not fully inside the circle it is crucial that you identify and mark the correct circle.
3.
  - a) Follow oral instructions to enter your name.
  - b) Enter your four-digit registration number. (This is the number on your 3 x 5 card.) Then blacken the corresponding numbers underneath.
  - c) Failure to correctly fill in your code number is grounds for disqualification.
4. Indicate your answers on the answer sheet. No credit will be given for answers written in this examination booklet. Be sure that each mark on the answer sheet is black and **completely fills the circle**. Give only one answer to each question. No credit will be given for multiple answers. If you change an answer, be sure that all previous marks are completely erased. Avoid accidental marks in any other space.
5. Your score will be the number of correct answers marked minus one fourth of the number incorrect answers marked. This is to discourage guessing.
6. Do not spend too much time on any one question. Answer the easier questions first and then go back to the others if time permits.
7. Many questions, even those which look unfamiliar to you, require no specialized mathematical knowledge beyond 9th grade algebra. The questions are not in order of difficulty. If you get stuck, go on; there may be easier questions for you further on.
8. There are 2 blank pages at the end for scratch work.
9. **Begin the test only when told to do so.**
10. You may take the questions home with you.
11. Check the following carefully:
  - a) Name is encoded according to the oral instructions.
  - b) Your 4-digit registration number is coded. (This is the number on your 3 x 5 card.)
  - c) Only one answer per question.
  - d) There are 67 problems typed on 14 pages. The test is double-sided.

1. What is the value of  $\frac{1 - \left(\frac{3}{5} - \frac{2}{3}\right)}{2 - \frac{2}{9}}$  ?

- a)  $\frac{3}{5}$       b)  $\frac{6}{5}$       c)  $\frac{9}{8}$       d)  $\frac{3}{8}$       e)  $\frac{16}{9}$

2. The sum of the roots of  $x^2 - 5x - 36 = 0$  is

- a) 13      b) -13      c) 5      d) -5      e) 7

3. Bacteria in a petri dish double every day. If the whole dish is covered in 10 days, in how many days will half the petri dish be covered?

- a) 5      b)  $\log_{10}$       c) 12      d)  $\sqrt{10}$       e) 9

4. Six couples have a dinner party. In how many ways can they be seated evenly apart at a circular table so that males and females alternate and each man is opposite his own wife.

- a) 6!      b) 5!      c)  $2 \cdot 5!$       d) 0      e) None listed.

5. Compute

$$(1 + 2i)(1 + 3i)(1 - 3i)(1 + i)(2 + i)(4 - 2i)(1 - i)(1 - 2i)$$

- a) 1000      b)  $3 + 4i$       c)  $100i$       d)  $48 - 24i$       e) None listed.

6. If the centers of spheres of radius 1 are placed at each of the eight vertices of a cube of side 2, how much of the volume of the cube is not contained in any of the spheres?

- a)  $1 - \frac{2\pi}{3}$       b)  $8 - \frac{4\pi}{3}$       c)  $4 - \frac{4\pi}{3}$       d)  $8 - \frac{2\pi}{3}$       e)  $4 - \pi$

7. If  $\cos \theta = \frac{1}{2\sqrt{3}}$ , then  $\sin^2 \theta + \frac{1}{1 + \tan^2 \theta}$  is which of the following?

- a) 0      b) 1      c) 2      d)  $\frac{1}{2}$       e) None listed.

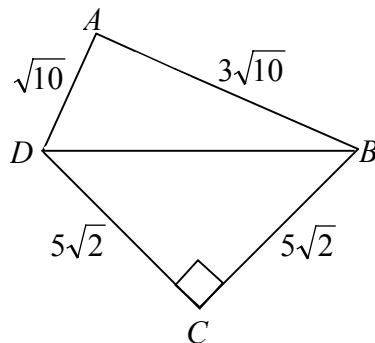
8. Compute the sum  $(1-1/3)+(1/2-1/4)+(1/3-1/5)+\dots+(1/171-1/173)$ . Express your answer to the nearest thousandths.

- a) 0.488      b) -0.512      c) 4.488      d) 1.488      e) -1.512

9. The co-ordinates of two corners of a square are (3,6) and (3,10). The area of the square could be

- a) 2 or 8      b) 4 or 8      c) 8 or 16      d) 16 or 20      e) None listed.

10. Compute  $AC$



- a) 10      b) 8      c)  $5\sqrt{5}$       d)  $4\sqrt{5}$       e)  $4\sqrt{10}$

11. In a city,  $\frac{2}{3}$  of the adult men are married to  $\frac{3}{5}$  of the adult women. The number of married men and women are equal, and the adult population is over 1500. What is the smallest possible number of adult residents in the city?

- a) 1501      b) 1510      c) 1519      d) 1528      e) None listed.

12. Assume that  $x - \frac{1}{x} = a$ . What is the value of  $x^3 - \frac{1}{x^3}$ ?

- a)  $a^3 - 3a$       b)  $a^3 + 3a$       c)  $a^3 - 3a^2 + 3a$       d)  $a^3 + 3a^2 - 3a$       e) None listed.

13. Two trains start from stations that are 450 miles apart, at the same time, and travel towards each other. The speed of the first train is twice that of the other. If they meet after 5 hours, the speed of the first train is

- a) 30 mph      b) 45 mph      c) 60 mph      d) 75 mph      e) 90 mph

14. If  $0 \leq x \leq y \leq 3$ , the smallest possible value of  $\left(x + \frac{1}{y}\right)^2$  is

- a) 9      b) 3      c)  $\frac{1}{9}$       d)  $\frac{1}{3}$       e) None listed.

15. Find the value of  $\sin 70^\circ \cos 50^\circ + \sin 230^\circ \cos 250^\circ$ .

- a)  $\frac{\sqrt{3}}{2}$       b)  $\frac{\sqrt{3}}{4}$       c)  $\sqrt{3}$       d)  $\frac{\sqrt{2}}{2}$       e)  $\frac{\sqrt{3}}{2\sqrt{2}}$

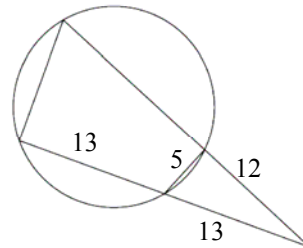
16. What is the area of the region in the first quadrant defined by  $20 \leq x + y \leq 24$ ?

- a) 88                      b) 288                      c) 400                      d) 480                      e) None listed

17. The number of pairs of integers  $(x, y)$  satisfying  $x^2 = 3y + 2$  is

- a) 0                      b) = 1                      c) 19                      d) 23                      e) infinite

18. Find the area of the quadrilateral.



- a) 65                      b) 120                      c)  $\frac{845}{6}$                       d)  $\frac{845}{12}$                       e)  $\frac{665}{6}$

19. The mean of two numbers is  $m$ . One of the two numbers is  $x$ . The other number is

- a)  $m - x$                       b)  $x - m$                       c)  $\frac{xm}{2}$                       d)  $2m - x$                       e)  $m - 2x$

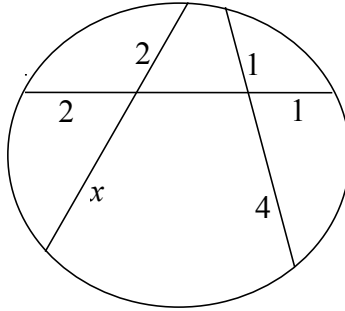
20. If  $0 \leq x \leq 3y$  and  $(x + y)^2 - (x - y)^2 \geq 108$ , what is the least possible value of  $y$ ?
- a) 0                      b) 3                      c) 6                      d) 9                      e) None listed.
21. A fair coin is tossed six times. The probability of getting exactly 2 heads is
- a)  $\frac{6}{64}$                       b)  $\frac{15}{64}$                       c)  $\frac{32}{64}$                       d)  $\frac{40}{64}$                       e) None listed.
22. When John divides his favorite number by 11, he gets a remainder of 4. What will the remainder be if he multiplies his favorite number by 4 and then divides by 11?
- a) 11                      b) 6                      c) 5                      d) 4                      e) 2
23. A convex polygon has 65 distinct diagonals. How many sides does it have?
- a) 16                      b) 15                      c) 14                      d) 13                      e) 12
24. Nate's strategy for a 35.1 mile race is to run 2 miles, walk 1 mile, run 2 miles, walk 1 mile, and continue this pattern until he completes the race. Nate's average running pace is 8 minutes per mile and his average walking pace is 16 minutes per mile. How many minutes will it take Nate to complete the race? Express your answer to the nearest tenth.
- a) 370.6                      b) 371.6                      c) 366.6                      d) 372.6                      e) 369.6
25. Suppose an isosceles triangle is formed with a tangent line to a circle of radius 2 and the radius to this circle. The hypotenuse is given by
- a) 2                      b) 4                      c)  $2\sqrt{2}$                       d)  $\sqrt{2}$                       e) Not enough information given

26. A box contains 100 chips numbered 1 through 100. Three chips are drawn randomly one after another without replacement. What is the probability that the number on the second draw is the largest of the three numbers drawn?
- a)  $\frac{1}{3}$                   b)  $\frac{1}{6}$                   c)  $\frac{1}{100}$                   d)  $\frac{1}{2}$                   e) None listed.
27. The length of a rectangle has increased by 60%. By what percent should the width be increased to double the area?
- a) 40%                  b) 20%                  c) 25%                  d) 30%                  e) 35%
28. A function  $f$  satisfies  $f(n+2) - f(n) = 11$  for all natural numbers  $n$ . Find  $f(5) - f(201)$ .
- a) -1078                  b) -2156                  c) -1089                  d) -1067                  e) -2145
29. If  $1 + \tan a + \tan b = \tan a \tan b$ , which of the following statements is correct about  $a + b$ ?
- a)  $\sin(a + b) = \pm \frac{\sqrt{2}}{2}$ .
- b)  $\sin(a + b)$  must be equal to  $\frac{\sqrt{2}}{2}$ .
- c)  $\sin(a + b) = \pm \frac{\sqrt{3}}{2}$ .
- d)  $a + b$  should lie in the first or the third quadrant.
- e) None of the above statements is true.
30. What is  $(1 + i)^8$ ?
- a) 16                  b)  $8 + 8i$                   c) -8                  d) 64                  e) None listed.

31. A point reflected first on the line  $x + y = 0$  and then on the  $x$ -axis. The resulting point has co-ordinates  $(-1, -3)$ . The co-ordinates of the original point is
- a)  $(3, 1)$       b)  $(-3, 1)$       c)  $(3, -1)$       d)  $(-3, -1)$       e) None listed.
32. John can finish a task in 12 days and Jack in 24 days. If they work together, in how many days can they finish the task?
- a) 6 days      b) 8 days      c) 9 days      d) 12 days      e) 18 days
33.  $\sin^4 15^\circ + \cos^4 15^\circ =$
- a) 1      b)  $\frac{31}{32}$       c)  $\frac{15}{16}$       d)  $\frac{7}{8}$       e)  $\frac{3}{4}$
34. If one selects the digit  $d$  from the set  $\{0, 1, 2, \dots, 9\}$ , then what is the probability that the number  $182d7$  is divisible by 7?
- a)  $\frac{1}{5}$       b)  $\frac{3}{10}$       c)  $\frac{2}{5}$       d)  $\frac{1}{3}$       e) None listed.
35. If the system  $\begin{cases} x + ay = 3 \\ 4x + 5y = 6 \end{cases}$  has no solution, then the value of  $a$  is
- a) 1      b)  $\frac{1}{4}$       c)  $\frac{1}{2}$       d)  $\frac{3}{4}$       e)  $\frac{5}{4}$
36. Let  $a*b=2310$ , where  $a$  and  $b$  are both two-digit numbers. What is the greatest possible value of  $a+b$ ?
- a) 107      b) 108      c) 109      d) 104      e) 110



37. Find  $x$ .



- a) 5                      b) 3                      c) 4                      d)  $\sqrt{5}$                       e) Not enough information.

38. Evaluate  $\lim_{n \rightarrow \infty} \frac{1}{n} \left[ \left(\frac{1}{n}\right)^9 + \left(\frac{2}{n}\right)^9 + \left(\frac{3}{n}\right)^9 + \dots + \left(\frac{n}{n}\right)^9 \right]$ .

- a)  $\frac{1}{e^9}$                       b) 0                      c)  $\frac{1}{\ln 9}$                       d)  $\frac{1}{10}$                       e) None listed.

39. If  $\frac{\frac{x}{1} - \frac{y}{1}}{\frac{1}{x} + \frac{1}{y}}$  is simplified you get

- a) 0                      b)  $\frac{x-y}{x+y}$                       c)  $\frac{x+y}{x-y}$                       d)  $\frac{x+y}{y-x}$                       e)  $\frac{y-x}{x+y}$

40. Given  $P(x)$  is a polynomial such that  $P(x^3 - 1) = x^6 + 4x^3 + 5$ . Then  $P(x^3 + 1)$  is given by

- a)  $x^6 + 8x^3 + 13$     b)  $x^6 + 4x^3 + 5$     c)  $x^6 + 8x^3 + 17$     d)  $x^6 + 4x^3 + 13$     e) None listed.

41. Define if  $\log_{10} x = \log x$ . What is the value of  $\log \frac{1}{2} - \log \frac{3}{2} - \log \frac{4}{3} - \dots - \log \frac{99}{98} - \log \frac{100}{99}$  ?

- a) 1                      b) 0                      c) -1                      d) -2                      e) None listed.

42. The line  $y = bx + b$  intersects the parabola  $y = x^2$  in exactly one point. If  $b \neq 0$ , then  $b$  should be

- a) -5                      b) -4                      c) -3                      d) -2                      e) -1

43. What is the greatest number of bags that can be used to hold 66 marbles if each bag must contain one marble but no two bags may contain the same number of marbles and you cannot put any bag inside any other bag?

- a) 10                      b) 11                      c) 12                      d) 13                      e) 9

44. NASA wishes to place six satellites in orbit around a spherical moon, so that a rover on the moon's surface can always transmit to at least one of them.

What is the minimum height from the surface of the moon at which the satellites can be placed, if the radius of the moon is  $r$ ?

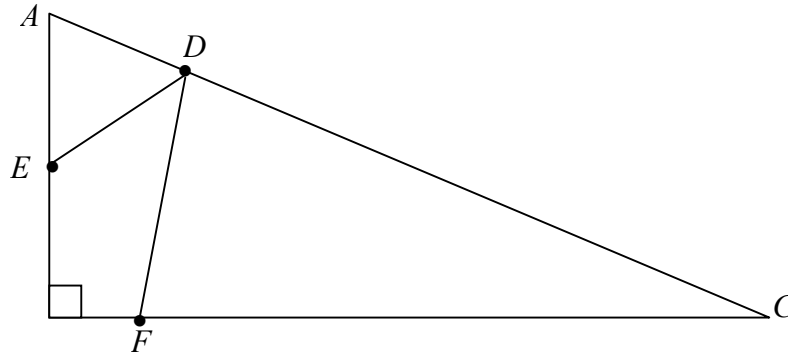
- a)  $r$                       b)  $r\sqrt{2}$                       c)  $r(\sqrt{2} - 1)$                       d)  $r(\sqrt{3} - 1)$                       e)  $r\frac{\sqrt{3}}{2}$

45. If  $0 < x < \frac{\pi}{2}$  and  $\log_{\cos x} \sin x = \frac{1}{2}$ , then the value of  $\cos x$  is

- a)  $\frac{\sqrt{5}-1}{2}$                       b)  $\frac{1}{\sqrt{3}}$                       c)  $\frac{2}{\sqrt{3}}$                       d)  $\frac{\sqrt{3}-1}{2}$                       e) None listed.

46. For a quadratic function  $f(x) = ax^2 + bx + c$ , if  $c = \frac{b^2}{4a}$ , then the graph of  $y = f(x)$  will
- a) have a minimum      b) have a maximum      c) be tangent to the  $x$ -axis      d) be tangent to the  $y$ -axis      e) None listed.
47. A pair of dice is rolled. If one die shows a “six”, what is the probability of “double six”?
- a)  $\frac{1}{6}$       b)  $\frac{1}{9}$       c)  $\frac{1}{11}$       d)  $\frac{1}{12}$       e) None listed.
48. Suppose  $x > 0$ . What is the value of  $\cos\{\arctan[\sin(\operatorname{arccot} x)]\}$ ?
- a)  $\sqrt{\frac{x^2+1}{x^2+2}}$       b)  $\sqrt{\frac{x^2+2}{x^2+1}}$       c)  $\frac{1}{\sqrt{x^2+1}}$       d)  $\frac{1}{\sqrt{x^2+2}}$       e) None listed.
49. The vertices of square  $EFGH$  lie on the edges of a square  $ABCD$  such that  $AE/EB=BF/FC=CG/GD=DH/HA=1/2$ . What is the ratio of the area of square  $EFGH$  to the area of square  $ABCD$ ?
- a)  $\frac{4}{9}$       b)  $\frac{2}{3}$       c)  $\frac{5}{9}$       d)  $\frac{1}{3}$       e)  $\frac{7}{9}$
50. What is the greatest number of pieces you can cut a round cake into by making four straight cuts with a knife? Each cut must pass through the top and bottom of the cake.
- a) 9      b) 10      c) 11      d) 12      e) 13

51. DX 10. In the right triangle  $ABC$ ,  $AD = AE$  and  $CD = CF$ . Can you determine the angle  $\angle EDF$ ?



- a)  $15^\circ$       b)  $30^\circ$       c)  $37.5^\circ$       d)  $45^\circ$       e)  $60^\circ$
52. The sum of the digits of an odd two-digit prime number is 11. The tens digit is greater than the ones digit. What is the ones digit?
- a) 9      b) 7      c) 5      d) 3      e) 1
53. A pair of dice is rolled until the sum of the two numbers obtained is 7. What is the probability that a 7 is obtained at the 6<sup>th</sup> toss and not before? Express your answer to the nearest hundredth.
- a) 0.06      b) 0.05      c) 0.1      d) 0.07      e) 0.04
54. A regular hexagon  $ABCDEF$  has side length 11 inch. The number of square inches in the area of quadrilateral  $ABCE$  can be expressed as  $x \cdot \sqrt{3}$ . Find  $x$ .
- a) 100      b) 81      c) 196      d) 121      e) 64

55. What is the maximum value of the function  $f(x) = \frac{2\sin^3 x \cos x}{1 + \tan^2 x}$  ?

- a)  $\frac{1}{8}$       b)  $\frac{1}{4}$       c)  $\frac{1}{2}$       d)  $\frac{3}{4}$       e) 1

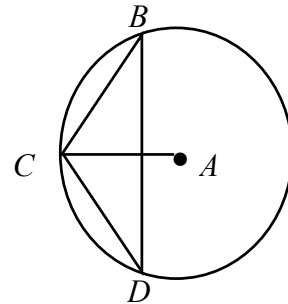
56. Given a square having vertices  $(-1,1)$ ,  $(1,1)$ ,  $(1,-1)$ , and  $(-1,-1)$ , find the probability that a randomly chosen point inside the square will have the sum of its coordinates greater than  $\frac{1}{9}$ .

- a)  $\frac{289}{810}$       b)  $\frac{289}{324}$       c)  $\frac{361}{486}$       d)  $\frac{289}{648}$       e)  $\frac{289}{567}$

57. A van holds exactly 8 people, but only 4 of these 8 people can drive the car. What is the number of ways that the 8 people can be seated in the car on a drive?

- a) 96      b) 20160      c) 40320      d) 15120      e) None listed.

58. In the circle with  $A$  as the center,  $\angle CBD = 15^\circ$ , then  $\angle ACD$  is



- a)  $75^\circ$       b)  $60^\circ$       c)  $45^\circ$       d)  $30^\circ$       e) None listed.

59. For each positive integer  $n$ , the mean of the first  $n$  terms of a sequence is  $n$ . What is the 2009<sup>th</sup> term of the sequence?

- a) 2009      b) 4016      c) 4017      d) 4,030,056      e) 4,032,064

60. Six couples have a cocktail party. In how many ways can they be seated at a circular table so that males and females alternate and each man is next to his own wife?

By convention, two arrangements are the same if each person has the same person on his or her left and right.

- a) 120      b) 240      c) 60      d) 720      e) 12!
61. Judy's favorite clothes include four T-shirts, three pairs of designer jeans, and two pairs of sandals. How many days in a row could she wear a different outfit using her favorite clothes?
- a) 12      b) 16      c) 20      d) 24      e) 28
62. If you got this far, you will now get a freebie. The correct answer is A. Math is
- a) lots of fun    b) fun      c) some fun    d) no fun      e) not at all fun
63. What is the remainder when  $P(x) = x^{2010} - x^{2008} + (x+1)^2$  is divided by  $x^2 - 1$ ?
- a) 2      b) 4      c)  $x+1$       d)  $2x+2$       e)  $x+3$
64. Trapezoid  $ABCD$  has vertices  $A(0,0)$ ,  $B(12,0)$ ,  $C(15,4)$ , and  $D(0,4)$ . A line through the origin intersects  $DC$  at  $E$ , dividing the trapezoid into two regions with equal perimeters. What is the area of quadrilateral  $ABCE$ ?
- a) 25      b) 24      c) 26      d) 29      e) 23

65. What is the value of  $\log_2\left((\log_{27} 3)^{\log_3 27}\right)$ ?

- a) 3                      b) -3                      c)  $3\log_2 3$                       d)  $-3\log_2 3$                       e) None listed.

66. The length of a rectangle is increased by 20% and the width by 50%. By what % will the area increase?

- a) 40                      b) 50                      c) 60                      d) 70                      e) 80

67. Suppose  $f$  is a continuous function that satisfies

$$\int_0^x f(t)dt = xe^{2x} - \int_0^x e^{-t} f(t)dt$$

for all  $x$ . Find an explicit formula for  $f(x)$ .

- a)  $\frac{(1+2x)e^{3x}}{e^x+1}$   
b)  $\frac{(1+2x)e^{3x}}{1+e^{-x}}$   
c)  $\frac{(1+2x)e^{2x}}{1+e^x}$   
d)  $(1+2x)e^{3x}$   
e) None listed.