1. Let $S(n, k)$ denote the Stirling number of the second kind. Show that

$$S(n, k) = \sum_{r=k-1}^{n-1} \binom{n-1}{r} S(r, k-1).$$

2. Find the Ramsey number $r(3, 4)$. Prove that your answer is correct.

3. Find the number of ways of painting the four faces $a, b, c, d$ of the pyramid below with two colors of paint, $x$ and $y$.

4. Find an explicit formula for the function $r$ defined by

$$r(0) = 0,$$
$$r(n) = 2r(n-1) + 2^n, n \geq 1.$$

5. Consider the statement, "If $P(2n)$ is true for all natural numbers $n$, and $P(n) \Rightarrow P(n + 1)$ for all natural numbers $n$, then $P(n)$ is true for all natural numbers $n$.” Is this a true statement? If not, can you find a simple modification to make it true?

6. The Fibonacci numbers are defined by

$$F_0 = F_1 = 1, F_{n+1} = F_n + F_{n-1} \text{ for } n > 1.$$
The generating function is

\[ f(x) = F_0 + F_1 x + \ldots + F_n x^n + \ldots. \]

Show that

\[ f(x) = \frac{1}{1 - x(1 + x)}. \]

Hence find an expression for \( F_n \) as a sum of binomial coefficients.

7. The set \( S \) contains \( n + 1 \) different positive integers, none greater than \( 2n \). Use the pigeonhole principle to prove that \( S \) contains two elements whose sum is \( 2n + 1 \).