Ph. D. Qualifying Examination

For this exam you may use the textbook, N. N. Lebedev, Special Functions and Their Applications, as a reference during the exam, but no other outside material may be used.

1. Compute

\[ \psi(1/3). \]

2. Show that

\[ |\Gamma(iy)| = \frac{\pi}{y \sinh(\pi y)}. \]

3. Compute

\[ \int_{0}^{+\infty} \frac{dt}{(1+t)^2 \sqrt{1+1/t}}. \]

4. Define the Gegenbauer polynomials by the generating function

\[ (1-2xt+t^2)^{-\gamma} = \sum_{n=0}^{+\infty} C_n^{\gamma}(x)t^n, \]

where \( \gamma > 0, -1 \leq x \leq 1, |t| < 1. \)

(a) Find the three-term recursion relation satisfied by the \( C_n^{\gamma}(x) \).

(b) Find the differential equation satisfied by the \( C_n^{\gamma}(x) \).

5. Let \( \{L_n^{\alpha}(x)\} \) be the set of Laguerre polynomials. Show that, for \( \text{Re}(\alpha), \text{Re}(\gamma) > 0 \), we have

\[ \int_{0}^{+\infty} x^{\alpha}e^{-\mu x}L_n^{\alpha}(x)dx = \frac{\Gamma(n+\alpha+1)}{\mu^{\alpha+1}n!} \left( 1-\frac{1}{\mu} \right)^n. \]
6. Show that, if $\text{Re}(\mu + \nu) > -1$, then
\[
J_{\mu}(z)J_{\nu}(z) = \frac{2}{\pi} \frac{\pi/2}{0} \int J_{\mu + \nu}(2z \cos \theta) \cos((\mu - \nu)\theta) d\theta.
\]

7. Show that if $\text{Re}(\nu) > -1$, then
\[
\int_{0}^{z} J_{\nu}(t) dt = 2 \sum_{n=0}^{\infty} J_{\nu+2k+1}(z).
\]

8. Show that
\[
I_{\nu}(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_{\nu}(z) = \frac{1}{z}.
\]
(Hint: think Wronskian.)

9. Show that
\[
2^{m} \frac{d^{m}}{dz^{m}}(J_{n}(z)) = \sum_{k=0}^{m} \binom{m}{k}(-1)^{m-k} J_{n+m-2k}(z).
\]

10. Show that if $n$ is a nonnegative integer, then
\[
Q_{n}(z) = \frac{1}{2} P_{n}(z) \log \frac{z + 1}{z - 1} - f_{n-1}(z),
\]
where $P_{n}(z)$ is the $n$th Legendre polynomial and $f_{n-1}(z)$ is a certain polynomial.

11. Let, for $0 \leq k < 1$,
\[
K(k) = \int_{0}^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^{2} \sin^{2} \varphi}}
\]
be the elliptic integral of the first kind. Show that
\[
P_{-1/2}(\cosh \alpha) = \frac{2}{\pi} \text{sech}(\alpha/2) K(\tanh(\alpha/2)).
\]