MATH 519 QUALIFYING EXAM

FALL 2016

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. For any **even** integer $m \geq 2$, let S_m be the symmetric group of degree m. Consider two subgroups of S_m, $H = A_m$, and $K = \langle (12 \cdots m) \rangle$. Show that

$$(HK)/H \simeq \mathbb{Z}_2.$$

2. Let G be a group. Recall that an automorphism $\psi : G \to G$ is said to be *inner* if there exists $g \in G$ such that $\psi(x) = gxg^{-1}$ for all $x \in G$. Prove that the set of inner automorphisms forms a normal subgroup of the group of all automorphisms.

3. Let p and q be distinct primes with $p < q$. Classify all groups of order pq^2 which have only one Sylow p-subgroup.

4. Prove that an infinite simple group cannot have a subgroup of finite index.

5. Let G be a group and let K, H be subgroups such that $K \triangleleft H$.

(a) Prove that H normalizes $C_G(K)$.

(b) If $H \triangleleft G$ and $C_H(K) = \{1\}$, prove that H centralizes $C_G(K)$.

6. Recall that a group G is *solvable* if there is a chain of subgroups

$$\{1\} = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_s = G$$

such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, $i = 0, 1, \ldots, s-1$.

Show that any group G of order 12 is solvable.

7. Let p be a prime number. Show that the ideal $(x^{p-1}+x^{p-2}+\cdots+x+1)$ generated by the polynomial $x^{p-1}+x^{p-2}+\cdots+x+1$ is a maximal ideal in $\mathbb{Q}[x]$.

8. Give an example of:

(a) A commutative ring that is not an integral domain.
(b) A Euclidean domain that is not a field.
(c) An integral domain that is not a UFD.
(d) A maximal ideal in $\mathbb{Z}[x]$.

Justify your answers.

9. Let R be a commutative ring with 1. Let I, J be two ideals of R. Consider a ring homomorphism

$$\phi : R \to R/I \times R/J$$

defined by $x \longmapsto (x+I, x+J)$. Show that ϕ is surjective if and only if $I+J=R$.

10. The Jacobson Radical $J(R)$ of a ring R is defined to be the intersection of all maximal ideals of R. Let R be a commutative ring with 1 and let $x \in R$. Show that $x \in J(R)$ if and only if $1-xy$ is a unit for all $y \in R$.