MATH 519 QUALIFYING EXAM

SPRING 2015

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1.

(a) Prove that a group of order 160 is not simple.

(b) Prove that a group of order pqr for p, q, r distinct primes is not simple.

2. Let G be a finitely generated group, and $n > 1$ an integer. Show that G has at most finitely many subgroups of index n.

3. A sequence of groups and group homomorphisms

$$N \xrightarrow{\alpha} G \xrightarrow{\beta} Q$$

is said to be *exact at* G if and only if the image of α is equal to the kernel of β. Let 1 denote the trivial group. A *short exact sequence* is a sequence of groups and group homomorphisms

$$1 \longrightarrow N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \longrightarrow 1$$

that is exact at N, G, and Q.

(a) Construct homomorphisms α and β so that

$$1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

is a short exact sequence.

(b) Let S_3 denote the symmetric group on 3 elements, and construct homomorphisms α and β such that

$$1 \longrightarrow \mathbb{Z}/3\mathbb{Z} \xrightarrow{\alpha} S_3 \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \longrightarrow 1$$

is a short exact sequence.

4. Let G be the group of upper triangular invertible matrices over $\mathbb{Z}_3$,

$$G = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mid a, c \in \mathbb{Z}_3^\times, b \in \mathbb{Z}_3 \right\}.$$

Then $|G| = 12$. Let

$$U = \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \mid x \in \mathbb{Z}_3 \right\}, \quad T = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{Z}_3^\times \right\}.$$

Then U and T are subgroups of G.

a) Prove that $G = U \rtimes T$.

b) Since $|G| = 12$, the group G is isomorphic to one of the following groups:

$$\mathbb{Z}_{12},\ \mathbb{Z}_6 \times \mathbb{Z}_2,\ A_4,\ D_{12},\ \mathbb{Z}_3 \rtimes \mathbb{Z}_4.$$

Which one? Justify your answer.

5. List, up to isomorphism, all finite abelian groups A satisfying the following two conditions:

(i) A is a quotient of $\mathbb{Z}^2$, and

(ii) A is annihilated by 18, i.e., $18a = 0$ for all $a \in A$.

Your list should contain a representative of each isomorphism class exactly once. How many groups are there?

6. Let G be a finite group and let $a, b \in G$ be two distinct elements of order 2. Show that the subgroup of G generated by a and b is a dihedral group.

(Recall that the dihedral group D_{2n} is given by

$$D_{2n} = \langle r, s \mid r^n = 1,\ s^2 = 1,\ rs = sr^{-1} \rangle.)$$

7. We say that $a \in \mathbb{Q}(\sqrt{3})$ is integral over $R = \mathbb{Z}[\sqrt{3}]$ if and only if there is a monic polynomial $p \in R[x]$ such that $p(a) = 0$. Show that there is $a \in \mathbb{Q}(\sqrt{3})$ which is integral over R but not an element of R.

8. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ be a sequence of ideals in $\mathbb{Z}[i]$. Show that there is n such that for all $m > n$ we have $I_m = I_{m+1}$.

9. Decide which of the following rings are Euclidean Domains, Principal Ideal Domains or Unique Factorization Domains. Justify your answer.

(a) $\mathbb{Z}[x, y]$
(b) $\mathbb{Z}_5[x]$
(c) $\mathbb{Z}_6[x]$

10. Let R be a commutative ring with 1, and suppose I and J are ideals of R so that $I + J = R$. Show that

(a) $IJ = I \cap J$
(b) $R/IJ \cong R/I \oplus R/J$