Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let $G$ be a group with cyclic automorphism group $Aut(G)$. Prove that $G$ is abelian.

2. Prove that every group of order 35 is cyclic.

3. Show that if the center of a group $G$ is of index $n$ in $G$, then every conjugacy class of $G$ has at most $n$ elements.

4. Classify all abelian groups of order 120 and list at least 3 non-abelian groups of order 120. Prove that the non-abelian groups you listed are nonisomorphic.

5. Let $H$ be a proper subgroup of a finite group $G$. Prove that the union of the conjugates of $H$ is not the whole group $G$.

6. Let $N$ be a normal subgroup of the finite group $G$. A subgroup $H$ is said to be a complement for $N$ in $G$ if $NH = G$ and $N \cap H = 1$.
   (a) Show that all complements for $N$ in $G$ are isomorphic.
   (b) If $H$ is a complement for $N$ in $G$, show that any conjugate of $H$ is also a complement for $N$ in $G$.
   (c) If $N$ has a complement in $G$ that is a $p$-group for some prime $p$, prove that every Sylow $p$-subgroup of $G$ contains a complement for $N$. 
7. Let
\[ S = \{ \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \}. \]

(a) Prove that \( S \) is a subring of the ring \( M_2(\mathbb{Z}) \) of \( 2 \times 2 \)-matrices with coefficients in \( \mathbb{Z} \).

(b) Prove that the map \( \varphi : \mathbb{Z}[\sqrt{2}] \to S \) defined by
\[ \varphi(a + b\sqrt{2}) = \begin{pmatrix} a & b \\ 2b & a \end{pmatrix} \]
is a ring homomorphism.

8. An element \( e \) of a ring \( S \) is called idempotent if \( e^2 = e \). Note that in a product \( R \times R' \) of rings with unity, the element \( e = (1, 0) \) is idempotent. The object of this problem is to prove a converse for commutative rings.

(a) Prove that if \( e \) is an idempotent element of a ring with unity, then \( e' = 1 - e \) is also idempotent.

(b) Let \( e \) be an idempotent element of a commutative ring with unity, \( S \), and let \( e' = 1 - e \). Prove that \( S \) is isomorphic to the product \( (eS) \times (e'S) \).

9. Let \( R \) be a commutative ring, and let \( I \) be an ideal of the polynomial ring \( R[x] \). Suppose that the lowest degree of a nonzero element of \( I \) is \( n \) and that \( I \) contains a monic polynomial of degree \( n \). Prove that \( I \) is a principal ideal.

10. Let \( R \) be a commutative ring with 1 and \( M \) a maximal ideal of \( R \). Suppose \( I, J \) are ideals of \( R \) such that \( IJ \subseteq M \). Show that \( I \subseteq M \) or \( J \subseteq M \).