MATH 519 QUALIFYING EXAM

SPRING 2009

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Classify all groups of order $11^2 \cdot 17^2 \cdot 19^2$. (*Hint: First prove that any group of this order is abelian. You may use the fact that any group of order p^2 with p prime is abelian.*)

2. A group G is **solvable** if there is a chain of subgroups

$$1 = G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_s = G$$

such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, $i = 0, 1, \ldots, s-1$.

Let G be a group and N a normal subgroup. If N and G/N are solvable, prove that G is solvable.

3. Let $\mathbb{F}_q$ denote the finite field with q elements. Prove that

$$|GL(n, \mathbb{F}_q)| = (q^n - 1) \cdot (q^n - q) \cdot \cdots \cdot (q^n - q^{n-1}).$$

(*Hint: use induction, and consider the orbit and stabilizer of*

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

for the action of $GL(2, \mathbb{F}_q)$ on $\mathbb{F}_q^n$.)

4. Let G be a finite simple group. Assume that G is not prime. Prove that $|G|$ is divisible by two distinct primes, and by the square of a prime.

5. Prove that $\mathrm{Aut}(G_1 \times G_2) \cong \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2)$ whenever G_1 and G_2 are nonisomorphic simple groups. Give counterexamples when the hypotheses "simple" and "nonisomorphic" are omitted.

6. Prove that if p is prime and P is a subgroup of S_p of order p, then $|N_{S_p}(P)| = p(p-1)$.

(Hint. Argue that every conjugate of P contains exactly $p-1$ p-cycles. Then compute the total number of p-cycles, and use this to determine the index of $N_{S_p}(P)$ in S_p.)

7. Let R be a non-zero commutative ring with 1 and S subset of R which is closed under multiplication and does not contain 0. Show that if P is maximal in the set of ideals of R not intersecting S, then P is a prime ideal.

8. Let F be a field. Prove that the ring $F[x, y]$ of polynomials in two variables over F is not a Euclidean domain.

9. Suppose that R is a commutative ring with $1 \neq 0$, and that $|R/I| < \infty$ for every ideal I of R. Prove that an ideal I of R is prime if and only if it is maximal.

10. Let R be the ring of all continuous functions from the closed interval $[0, 1]$ to $\mathbb{R}$. For $c \in [0, 1]$, let M_c be the ideal

$$M_c = \{f \in R \mid f(c) = 0\}$$

(a) Prove that M_c is a maximal ideal of R.
(b) Prove that M_c is not equal to the principal ideal generated by $x - c$.
(c) Is M_c principal?