MATH 519 QUALIFYING EXAM

Spring, 2010

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Prove no group of order 30 is simple.

2. Let G be a group, and H a normal subgroup such that $G/H \cong A_4$. Prove that G has a normal subgroup of index 3.

3. Let G be an abelian group of order $p^a m$, where p is a prime, $a \geq 1$ and $(p, m) = 1$.

(a) Prove G has a subgroup M of order m.
(b) Prove M is characteristic, that is, given an automorphism σ of G, show that $\sigma(M) \subset M$.

4. Suppose that a group G has a normal subgroup $K \not\leq Z(G)$ with 17 elements. Prove that $|G|$ is even.

5. View S_4 as the permutations of $\{1, 2, 3, 4\}$ and S_5 as the permutations of $\{1, 2, 3, 4, 5\}$. Let $S_4 \times S_4$ act on S_5 by $(\sigma_1, \sigma_2) \cdot \tau = \sigma_1 \tau \sigma_2$. Prove the orbit of $(1\ 5)$ is $S_5 \setminus S_4$.

6. A group G is said to be *divisible* if, for every $x \in G$ and every $n \in \mathbb{Z}^+$ there exists $y \in G$ such that $x^n = y$. Prove that every nontrivial divisible group is infinite.

7. Let R and S be commutative rings. Let $\varphi : R \to S$ be a surjective homomorphism. Let I be an ideal of R that contains $\ker(\varphi)$.

(a) Prove $\varphi(I)$ is an ideal of S.
(b) Prove $R/I \cong S/\varphi(I)$.

8. Let F be a field, and consider the polynomial ring $F[X, Y]$ in two indeterminates over F. Prove that the ideal $(X - 1, Y + 1)$ is maximal and the ideal $(X - Y)$ is prime. Is the ideal $(X^2 - Y^2)$ also prime? Justify your answer.

9. Let R be a UFD. Show that every non-zero prime ideal of R contains a non-zero principal ideal that is also prime.

10. Prove that $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ is a principal ideal domain.