MATH 505 — Qualifying Exam

Dept. of Mathematics
Southern Illinois University Carbondale

Spring, 2012
Jan. 24, 4:30-8:30p.m.

Instructions:

1. Write your name on this answer booklet.
2. Read each question carefully.
3. Please write legibly.
4. TO ENSURE FULL CREDIT, EXPLAIN YOUR WORK FULLY.
5. This exam has 10 pages.
6. You are only required to solve *six* problems.
7. Books and notes are not allowed in this exam.
8. Independent work is expected.
9. Good luck!

Name: ____________

Student ID: ____________

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________
8. ____________
9. ____________

Total:

1. Show that the system

$$\frac{dx}{dt} = x - y - x(x^2 + y^2)$$
$$\frac{dy}{dt} = x + y - y(2x^2 + y^2)$$

has a periodic solution. Hint: Rewrite the system in polar coordinates and then construct a trapping region (i.e. the state (x, y) will enter into this region at some time $t > 0$ and stay there afterwards).

2. Consider the system

$$\dot{x}(t) = f(t)x(t),$$

where f is a continuous function with period 1. Compute the *time-1* map of this system, i.e. the function $F_1(x)$ which has the property that $F_1(\xi) = x(1)$, where $x(t)$ solves

$$\dot{x}(t) = f(t)x(t), \qquad x(0) = \xi.$$

From this, compute a formula for $x(n)$ in terms of f and $x(0)$ for any integer n. What is a sufficient condition on f so that

$$\lim_{t\to\infty} x(t) = 0?$$

3. Show that the system

$$\dot{x} = -x + 2y$$
$$\dot{y} = -x - 2y^3$$

has no periodic orbits. Hint:Show that the function $V(x, y) = ax^2 + by^2$ is a strict Lyapunov function for this system for some values of a, b. Determine at least one such pair (a, b). Deduce from that there are no periodic orbits.

4. Let $f(x) = x^{2/3}\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$.

(a) Decide whether the initial value problem $dx/dt = f(x)$, $x(0) = 0$ has a unique solution. If the solution is unique, then prove it. If not, show it.

(b) What condition does f satisfy in a neighborhood of the origin?

(c) Can any solution of the initial value problem with $x(0) \neq 0$ become unbounded in x as t increases?

(d) Are solutions of the initial value problem with $x(0) \neq 0$ defined for all time t?

5. Consider the equation $\dot{x} = Ax$, where $x \in R^2$ and A is an 2-by-2 real constant matrix. Let λ_1, λ_2 be the eigenvalues of A. Assume that both eigenvalues have zero real parts $Re(\lambda_1) = Re(\lambda_2) = 0$.

(a) Is the equilibrium $x = 0$ stable for any such A? If yes, then prove it. If no, then give a counterexample.

(b) If the answer to the above question is no, then what additional conditions must be imposed on A so that the equilibrium point $x = 0$ is Lyapunov stable. Justify your answer.

6. Consider the initial value problem (IVP)

$$(1-2t)x'(t)+\alpha_1 x(t)=\alpha_2, \qquad x(0)=A, \tag{1}$$

where α_1, α_2, A are real constants and $\alpha_1>0$. We are aiming to show that this IVP has infinitely many solutions for $t\in[0,1]$.

(a) Show that the IVP has a unique solution for $t\in[0,1/2)$. Determine this solution.

(b) Show that

$$(1-2t)x'(t)+\alpha_1 x(t)=\alpha_2, \qquad x(1)=B,$$

has a unique solution for $t\in(1/2,1]$, where B is a real constant.Determine this solution.

(c) Show that the boundary value problem

$$(1-2t)x'(t)+\alpha_1 x(t)=\alpha_2, \qquad x(0)=A, \quad x(1)=B$$

has a continuous solution for any real B, and compute this solution. (From this it follows that the original IVP (1) has infinitely many solutions, one for each B.)

7. Consider the system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 6 \end{pmatrix},$$

(a) Find the stable, unstable, and center subspaces E^s, E^u, and E^c for this system;
(b) For $x_0 \in E^c$, show that the sequence $x_n = e^{An}x_0 \in E^c$ for $n = 1, 2, \ldots$.
(c) Solve the system.
(Recall that the stable subspace, which is spanned by the generalized eigenvectors corresponding to the eigenvalues λ with $\mathrm{Re}\lambda < 0$; the unstable subspace, which is spanned by the generalized eigenvectors corresponding to the eigenvalues λ with $\mathrm{Re}\lambda > 0$; the center subspace, which is spanned by the generalized eigenvectors corresponding to the eigenvalues λ with $\mathrm{Re}\lambda = 0$.)

8. Consider the equation

$$\frac{dx}{dt} = x^{\alpha}, \qquad t \geq 0, \qquad x \geq 0.$$

(a) For which $\alpha \in [0, \infty)$, does the solution have a unique solution with $x(0) = 0$?
(b) For which $\alpha \in [0, \infty)$ can any solution be extended to all time t in $[0, \infty)$?
(c) Explain your answers. Either refer to appropriate theorems or prove your conclusions.

9. Consider the ODE

$$\dot{x} = x \ln\left(\frac{1}{|x|}\right),$$

where the right hand-side is defined to be zero at $x = 0$. Consider the initial value problem $x(0) = 0$ which has a solution $x(t) = 0$.

(a) Does the uniqueness theorem apply here?

(b) If not, can you determine if this solution is unique?