Math 507 — Qualifying Exam

Partial Differential Equations — Spring, 2012

Southern Illinois University Carbondale — Jan. 24, 4:30-8:30p.m.

Instructions:

1. Read each question carefully.
2. Please write legibly.
3. You only need to solve **six** problems.
4. TO ENSURE FULL CREDIT, EXPLAIN YOUR WORK FULLY.
5. This exam has 12 pages, including this cover page.
6. Independent work is expected.
7. Good luck!

Name: ____________

ID: ____________

1. ____________

2. ____________

3. ____________

4. ____________

5. ____________

6. ____________

Total:

1. Solve the wave equation

$$
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0, \qquad 0 < x < \infty, \qquad t > 0,\\
&u(x,0) = u_t(x,0) = 0, \qquad x > 0,\\
&u_x(0,t) = \cos(t), \qquad t > 0.
\end{aligned}
$$

2. Let $\Omega_T = \{(x,t) | 0 < x < l, 0 < t \le T\}$. Let $u \in C(\bar{\Omega}_T) \cap C^{2,1}(\Omega_T)$ be the solution of

$$u_t - u_{xx} = 0, \quad (x,t) \in \Omega_T$$

satisfying $u(x,0) = 0, 0 \le x \le l$ and $u_x(0,t) + h(u_0 - u(0,t)) = 0, u(l,t) = 0$ for $0 \le t \le T$. Here h, u_0 are positive constants. Show that

$$0 \le u(x,t) \le u_0, \qquad (x,t) \in \Omega_T.$$

3. Let $B = \{x \in R^2 \mid |x| < 1\}$. Show that if $u \in C^2(B) \cap C^0(\bar{B})$, $u(x) = 0$ for $|x| = 1$. and $|\Delta u| \leq K$, where $K > 0$ is a constant, then we have

$$-\frac{K}{4} \leq u \leq \frac{K}{4}.$$

Hint: Use maximum principle for a function $v = u - w$ where $w(x) = 0$ for $|x| = 1$, $\Delta w = \pm K$. Note that w is a simple polynomial.

4. We wish to solve the heat equation $u_t - u_{xx} = 0$ for $x \geq 0$ and $t \geq 0$ with initial condition $u(x, 0) = 0$ and boundary condition $u(0, t) = Ct^n$, $t > 0$, where n is a positive integer and C is a positive constant. Use scaling of the dependent and independent variables to derive the representation $u = Ct^n f(\theta)$. What is θ? What boundary value problem does $f(\theta)$ satisfy?

5. Solve Laplace's equation $\Delta u = 0$ inside the unite circle with boundary condition $u(1, \theta) = f(\theta)$ at $r = 1$ where r is the radial distance to the center of the disc and θ is the polar angle, to derive Poisson's integral formula

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - \varphi)} d\varphi.$$

Recall that the Laplacian in polar coordinates is

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}.$$

6. Show that the solution of initial value problem $yu_x(x,y) - xu_y(x,y) = 0$ containing the curve $x^2 + y^2 = a^2$, $u = y$, does not exist.

7. The Green's function for a disk $B_R = \{x, |x| \leq R\}$ is given by

$$G(x; y) = \frac{1}{2\pi}\left[\ln\left(|\xi - y|\frac{|x|}{R}\right) - \ln|x - y|\right]$$

where $\xi = \frac{R^2}{|x|^2}x$ and $x \neq 0$. Show that for any $x, y \in \bar{B}_R$ with $x \neq y$, we have $G(x; y) \geq 0$ and $G(x; y) = 0$ if either $x \in \partial B_R$ or $y \in \partial B_R$.

8. Classify the PDE

$$2u_{xx} - 3u_{xy} + u_{yy} = y$$

as either hyperbolic, parabolic, or elliptic, and transform the equation to canonical form.

9. For the equation

$$u = xu_x + yu_y + \frac{1}{2}(u_x^2 + u_y^2)$$

find a solution with $u(x, 0) = \frac{1}{2}(1 - x^2)$.

10. Let $\Omega = \{(x, y) \in I\!R^2 \mid 0 < x^2 + y^2 < 1\}$. Prove there is no solution to the Dirichlet problem $\Delta u = 0$ in Ω, $u(x, y) = 1$ for $x^2 + y^2 = 1$, $u(0, 0) = 0$.

11. Let $u(x, t)$ be a solution of class C^2 of

$$u_t = a(x, t)u_{xx} + 2b(x, t)u_x + c(x, t)u$$

in the rectangle

$$\Omega = \{(x, t)|0 \leq x \leq L, \quad 0 \leq t \leq T\}.$$

Let $\partial'\Omega$ denote the "lower boundary" of Ω consisting of the three segments (i) $x = 0, 0 \leq t \leq T$, (ii) $0 \leq x \leq L, t = 0$, and (iii) $x = L, 0 \leq t \leq T$.

(1) Prove that in the case $c < 0$ in Ω

$$|u(x, t)| \leq \sup_{\partial'\Omega} |u| \quad \text{for } (x, t) \in \Omega.$$

(2) Show that more generally

$$|u(x, t)| \leq e^{CT} \max_{\partial'\Omega} |u|,$$

where

$$C = \max\left(0, \max_{\Omega} |c|\right).$$