MATH 520 QUALIFYING EXAM

FALL 2011

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Determine which of the following fields K are finite Galois extensions of $\mathbb{Q}$. For the ones that are Galois, find $Gal(K/\mathbb{Q})$.

a) $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})$ b) $\mathbb{Q}(\sqrt[3]{3})$

$c)$ $\mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/3} = \dfrac{-1+i\sqrt{3}}{2}$ is a primitive third root of unity.

2. Determine the splitting field of the polynomial $x^p - x - a$ over $\mathbb{F}_p$, where $a \neq 0$, $a \in \mathbb{F}_p$. Show explicitly that the Galois group is cyclic.
(*Hint.* Let α be a root. Show $\alpha + 1$ is root.)

3. Let $(m, n) = 1$ and let ζ_j denote a complex primitive j-th root of unity for any positive integer j. Show that

$$\mathbb{Q}(\zeta_m) \cap \mathbb{Q}(\zeta_n) = \mathbb{Q}$$

and

$$\mathbb{Q}(\zeta_{mn}) = \mathbb{Q}(\zeta_m)\mathbb{Q}(\zeta_n).$$

4. The polynomial $p(x) = x^3 + x + 1$ is irreducible over $\mathbb{F}_2$. Let α be a root of $p(x)$ and set $K = \mathbb{F}_2(\alpha)$.

a) Find $[K : \mathbb{F}_2]$ and $|K|$.
b) Show α is a generator of $K^\times$ by computing α^i for $1 \leq i \leq 15$.

5. Suppose K/F is a Galois extension and let σ be an element of the Galois group.

a) Suppose $\alpha \in K$ is of the form $\alpha = \beta/(\sigma\beta)$, for some nonzero $\beta \in K$. Prove that $N_{K/F}(\alpha) = 1$.

b) Suppose $\alpha \in K$ is of the form $\alpha = \beta - \sigma\beta$, for some $\beta \in K$. Prove that $Tr_{K/F}(\alpha) = 0$.

6. Let $\overline{\mathbb{Q}}$ be a fixed algebraic closure of $\mathbb{Q}$. Let F be a maximal subfield of $\overline{\mathbb{Q}}$ not containing $\sqrt{2}$ (such a subfield exists by Zorn's lemma). Show that every finite extension of F is cyclic.

7. Let R be a ring with 1 and let M be a left R-module. An element $e \in R$ is called a *central idempotent* if $e^2 = e$ and $er = re$ for all $r \in R$. Let e be a central idempotent.

(a) Prove that eM and $(1-e)M$ are submodules of M.

(b) Prove that $M = eM \oplus (1-e)M$.

8. The ring of Gaussian integers

$$\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$$

is a free $\mathbb{Z}$-module. For M given below, show that M is a free $\mathbb{Z}$-module. Find a basis and the rank of M. Write $z = (1-i) \otimes (2+3i)$ as a linear combination of the elements of the basis.

(a) $M = \mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{Z}[i]$

(b) $M = \mathbb{Z}[i] \otimes_{\mathbb{Z}[i]} \mathbb{Z}[i]$

9. Let R be a Principal Ideal Domain, let M be a torsion R-module and let p be a prime in R. Prove that if $pm = 0$, for some nonzero $m \in M$, then $Ann(M) \subseteq (p)$.

(Here, $Ann(M)$ denotes the annihilator of M, the ideal of R defined by

$$Ann(M) = \{r \in R \mid rx = 0, \text{ for all } x \in M\}.)$$

10. Suppose that R is a commutative ring and that M and N are flat R-modules. Prove that $M \otimes_R N$ is a flat R-module.