Ph. D. Qualifying Examination in Statistics
Thursday, January 15, 2004

1) Let $X_1, ..., X_n$ be iid from a normal distribution with unknown mean μ and known variance σ^2. Let $\bar{X} = \frac{\sum_{i=1}^n X_i}{n}$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X})^2$.

a) Show that $\bar{X}$ and S^2 are independent

b) Find the limiting distribution of $\sqrt{n}(\overline{X}^3 - c)$ for an appropriate constant c.

2) Let $X_1, ..., X_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(x) = \sqrt{\frac{\sigma}{2\pi x^3}} \exp\left(-\frac{\sigma}{2x}\right)$$

where x and σ are both positive. Then $X_i = \dfrac{\sigma}{W_i}$ where $W_i \sim \chi_1^2$. Find the uniformly minimum variance unbiased estimator (UMVUE) of $\dfrac{1}{\sigma}$.

3) Let $X_1, ..., X_n$ be a random sample from the distribution with density

$$f(x) = \begin{cases} \frac{2x}{\theta^2}, & 0 < x < \theta \\ 0 & \text{elsewhere} \end{cases}$$

Let $T = max(X_1, ..., X_n)$. To estimate θ consider estimators of the form CT. Determine the value of C which gives the smallest mean square error.

4) Let $X_1, ..., X_n$ be independent identically distributed random variables with probability mass function

$$f(x) = e^{-2\theta}\frac{1}{x!}\exp[\log(2\theta)x],$$

for $x = 0, 1, \ldots,$ where $\theta > 0$. Assume that at least one $X_i > 0$.

a) Find the maximum likelihood estimator of θ.

b) What is the maximum likelihood estimator of $(\theta)^4$? Explain.

5) Let $X_1, ..., X_n$ be independent identically distributed random variables from a distribution with pdf

$$f(x) = \frac{2}{\sigma\sqrt{2\pi}} \frac{1}{x} \exp\left(\frac{-[\log(x)]^2}{2\sigma^2}\right)$$

where $\sigma > 0$ and $x \geq 1$.

a) What is the UMP (uniformly most powerful) level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma = 2$?

b) If possible, find the UMP level α test for $H_o : \sigma = 1$ vs. $H_1 : \sigma > 1$.

6) Suppose that $X_1, ..., X_n$ are iid $N(0, \sigma^2)$ where $\sigma(> 0)$ is the unknown parameter. With preassigned $\alpha \in (0, 1)$, derive a level α likelihood ratio test for the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against an alternative hypothesis $H_a : \sigma^2 \neq \sigma_0^2$ in the implementable form.