Ph.D. Qualifying Examination in Statistics
4:30–8:30 Thursday, January 26, 2006

1. Let $Y_n \sim \chi^2_n$.

   a) Find the limiting distribution of $\sqrt{n}\ \left(\frac{Y_n}{n} - 1 \right)$.

   b) Find the limiting distribution of $\sqrt{n}\ \left[\left(\frac{Y_n}{n} \right)^3 - 1 \right]$.

2. Let $X_1, ..., X_n$ be iid with one of two probability density functions. If $\theta = 0$, then

$$f(x|\theta) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   If $\theta = 1$, then

$$f(x|\theta) = \begin{cases} \frac{1}{2\sqrt{x}} & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the Maximum likelihood estimator of θ.

3. Suppose $X_1, ..., X_n$ are iid uniform observations on the interval $(\theta, \theta+1)$, $-\infty < \theta < \infty$. Let $X_{(1)} = min(X_1, ..., X_n)$, $X_{(n)} = max(X_1, ..., X_n)$ and $T(\mathbf{X}) = (\mathbf{X_{(1)}}, \mathbf{X_{(n)}})$.

   a) Show that $T(\mathbf{X})$ is minimal sufficient statistic.

   b) Let $R = X_{(n)} - X_{(1)}$. Show that the probablity density function of R is given by

$$h(r|\theta) = \begin{cases} n(n-1)r^{(n-2)}(1-r) & 0 \le r \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Hint: Let $M = \frac{X_{(1)} + X_{(n)}}{2}$, then find the joint probability density function of R and M.

   c) Show that the minimal sufficient statistic $T(\mathbf{X})$ is not complete.

4. Let $X_1, ..., X_n$ be iid exponential(λ) random variables with $E(X_i) = \lambda$.

   a) Find information number $I_1(\lambda)$.

   b) Find the Cramér Rao lower bound (CRLB) for estimating $\tau(\lambda) = \lambda^2$.

   c) Find the uniformly minimum variance unbiased estimator (UMVUE) of λ^2.

5. Suppose that $X_1, ..., X_{10}$ are iid Poisson with unknown mean λ. Derive the most powerful level $\alpha = 0.10$ test for $H_0 : \lambda = 0.30$ versus $H_1 : \lambda = 0.40$. (You can use the attached tables).

6. Let $Y_1, ..., Y_n$ be iid $N(\mu, \sigma^2)$ random variables where μ and σ^2 are unknown. Set up the likelihood ratio test for $Ho : \mu = \mu_o$ versus $H_A : \mu \neq \mu_o$.