Ph.D. Qualifying Examination in Statistics
4:30–8:30 Thursday, September 2, 2010

1. Suppose that $X_1, X_2, ..., X_n$ are independent identically distributed random variables from normal distribution with unknown mean μ and known variance σ^2. Consider the parametric function $g(\mu) = e^{2\mu-1}$.

   a) Derive the uniformly minimum variance unbiased estimator (UMVUE) of $g(\mu)$.

   b) Find the Cramer-Rao lower bound (CRLB) for the variance of an unbiased estimator of $g(\mu)$.

   c) Is the CRLB attained by the variance of the UMVUE of $g(\mu)$?

2. Let $X_1, ..., X_n$ be independent identically distributed random variables from a normal distribution with mean μ and variance σ^2.

   a) Find the approximate distribution of $1/\bar{X}$. Is this valid for all values of μ?

   b) Show that $1/\bar{X}$ is asymptotically efficient for $1/\mu$, provided $\mu \neq \mu^*$. Identify μ^*.

3. Let $Y_1, ..., Y_n$ be independent identically distributed (iid) random variables from a distribution with probability density function (pdf)

$$f(y) = \frac{1}{2\sqrt{2\pi}} \left(\frac{1}{\theta}\sqrt{\frac{\theta}{y}} + \frac{\theta}{y^2}\sqrt{\frac{y}{\theta}} \right) \frac{1}{\nu} \exp\left[\frac{-1}{2\nu^2} \left(\frac{y}{\theta} + \frac{\theta}{y} - 2 \right) \right]$$

   where $y > 0, \theta > 0$ is **known** and $\nu > 0$.

   a) Find the maximum likelihood estimator (MLE) of ν.

   b) Find the MLE of ν^2.

4. Let $Y_1, ..., Y_n$ be iid gamma($\alpha = 10, \beta$) random variables. Let $T = c\overline{Y}$ be an estimator of β where c is a constant.

   a) Find the mean square error (MSE) of T as a function of c (and of β and n).

   b) Find the value c that minimizes the MSE. Prove that your value is the minimizer.

5. Let $Y_1, ..., Y_n$ be iid from a distribution with pdf

$$f(y) = \ 2\ \tau\ y\ e^{-y^2}\ (1 - e^{-y^2})^{\tau - 1}$$

for $y > 0$ and $f(y) = 0$ for $y \leq 0$ where $\tau > 0$.

a) Find a minimal sufficient statistic for τ.

b) Is the statistic found in a) complete? Prove or disprove.

c) Suppose T is the statistic obtained in (a) and (b) above. Find $E(T)$ and var(T).

6. Suppose that X is an observable random variable with its pdf given by $f(x)$. Consider the two functions defined as follows: $f_0(x)$ is the probability density function of a Beta distribution with $\alpha = 1$ and $\beta = 2$ and and $f_1(x)$ is the pdf of a Beta distribution with $\alpha = 2$ and $\beta = 1$.

a) Determine the UMP level $\alpha = 0.10$ test for $H_0 : f(x) = f_0(x)$ versus $H_1 : f(x) = f_1(x)$. (Find the constant.)

b) Find the power of the test in a).

7. Let $X_1, ..., X_n$ be a random sample from a uniform $(0, \theta)$ distribution $(\theta > 0)$. Let $Y = \max(X_1, ..., X_n)$. We are interested in interval estimators of θ.

a) Consider an estimator of the form $[aY, bY]$, $1 \leq a < b$, where a, b are constants. Find the confidence coefficient of this estimator.

b) Consider another estimator of the form $[Y + c, Y + d]$, $0 \leq c < d$, where c, d are constants. Find the confidence coefficient of this estimator.

c) Which of these estimators would you choose and why?