Ph.D. Qualifying Examination in Statistics
January 25, 2011

1. Let $Y_1, ..., Y_n$ be independent and identically distributed (iid) from a distribution with probability density function

$$f(y) = \frac{2y}{\theta^2}$$

for $0 < y \leq \theta$ and $f(y) = 0$, otherwise.

a) Find the limiting distribution of $\sqrt{n}\ \left(\ \overline{Y} - c\ \right)$ for appropriate constant c.

b) Find the limiting distribution of $\sqrt{n}\ \left(\ \log(\ \overline{Y}\ ) -\ d\ \right)$ for appropriate constant d.

c) Find the method of moments estimator of θ^k.

2. Let $(X_1, X_2, ..., X_n)$ be a random sample from a population with pdf

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta$$

and $f(x|\theta) = 0$ for $x < \theta$. For this distribution, the minimal sufficient statistic is complete. Find the uniformly minimum variance unbiased estimator of θ.

3. Let $Y_1, ..., Y_n$ be independent identically distributed random variables with pdf (probability density function)

$$f(y) = (2 - 2y) I_{(0,1)}(y)\ \nu\ \exp[(1-\nu)(-\log(2y - y^2))]$$

where $\nu > 0$ and $n > 1$. The indicator $I_{(0,1)}(y) = 1$ if $0 < y < 1$ and $I_{(0,1)}(y) = 0$, otherwise.

a) Find a complete sufficient statistic.

b) Find the (Fisher) information number $I_1(\nu)$.

c) Find the Cramer Rao lower bound (CRLB) for estimating $g(\nu) = \frac{1}{\nu}$.

4. Let $X_1, ..., X_n$ be independent and identically distributed (iid) from a distribution with probability density function

$$f(x) = \frac{2x}{\theta^2}$$

for $0 < x \leq \theta$ and $f(x) = 0$, otherwise. Let $T = c\max(X_1, ..., X_n) = cX_{(n)}$ be an estimator of θ where c is a constant.

a) Find the mean square error (MSE) of T as a function of c (and θ and n).

b) Find the value of c that minimizes the MSE. Prove that your value is the minimizer.

5. Two independent and identically distributed measurements X_1, X_2 are made from a pdf $f(x|\theta)$ with $\theta > 0$, where

$$f(x|\theta) = \begin{cases} \frac{1}{2}\, \theta\, e^{-\theta x}, & \text{if } \ x \geq 0, \\ \\ \frac{1}{2}\, \theta^{-1}\, e^{x/\theta}, & \text{if } \ x < 0. \end{cases}$$

a) Find the maximum likelihood estimator of θ in terms of X_1, X_2.

b) Find the most powerful hypothesis test of $H_0 : \theta = 1$ versus the alternative $H_1 : \theta = 2$ of size 0.10, giving an equation to determine the rejection cutoff but not solving it explicitly.

c) Suppose the observed values satisfy $x_1 > 0$ and $x_2 > 0$. Justify that your test is or is not UMP versus $H_1 : \theta > 1$.

6. Let $(X_1, X_2, ..., X_n)$ and $(Y_1, Y_2, ..., Y_m)$ be two independent random samples from normal populations with parameters (μ_1, σ_1^2) and (μ_2, σ_2^2), respectively. Assuming $\sigma_1^2 = 4\sigma^2$, $\sigma_2^2 = \sigma^2$, σ^2 unknown, develop a test of level α for all $\sigma^2 > 0$ when testing $H_0 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$.