Ph.D. Qualifying Exam in Topology

August 2004

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page number the pages. Do not use the back of a page. Good luck!

1. (Answer 5 of the 7, for 2 points each.)
   a) Define cardinality.
   b) State the Axiom of Choice.
   c) State the Maximum Principle.
   d) Define metric space.
   e) Define quotient map.
   f) Define strong deformation retract.
   g) Define covering map.

2. Construct an example showing that a quotient space of a Hausdorff space need not be Hausdorff.

3. Prove that $\mathbb{Q}$, the set of rational numbers with the usual topology, is not locally compact.

4. Show that every uncountable subset of the real line has a limit point.

5. Prove that every second countable space is both Lindelöf and separable.

6. Let X be a set and $\mathcal{S}$ a collection of subsets of X. Find necessary and sufficient conditions for $\mathcal{S}$ to be a sub-basis for a topology for X.

7. Prove that every compact Hausdorff space is normal.

8. Prove that X is Hausdorff if and only if the **diagonal** $\Delta := \{(x, x) \mid x \in X\}$ is closed in $X \times X$.

9. What is the fundamental group of the 2-dimensional sphere with two points deleted? Explain your reasoning.

10. Prove that if $p : E \longrightarrow B$ is a covering map, then p is an open map.