Ph.D. Qualifying Exam in Topology
January 2008

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Good luck!

1. (0-10 points; two points off for each wrong answer.)

   a) What is a Lindelöff space?

   b) Define local compactness.

   c) Define the compactification of a space.

   d) State the Lebesgue Number Lemma.

   e) State Tychonoff's Theorem.

   f) State the Tietze Extension Theorem.

   g) Let $f : X \to Y$ and $g : X \to Y$ be continuous maps between topological spaces. Define what it means for f to be homotopic to g.

2. (10 points) Construct an example showing that a quotient space of a Hausdorff space need not be Hausdorff.

3. (10 points) Let $I = [0, 1]$. Let $\mathcal{T}_p$ be the product topology for $I \times I$ and let $\mathcal{T}_d$ be the dictionary order topology on $I \times I$. Compare them.

4. (10 points) Let $X \subset Y \subset Z$, where Z is a topological space and Y has the subspace topology. Prove that the closure of X with respect to Y is the same as the closure of X with respect to Z.

5. (10 points) Let $\mathbb{R}_f$ be the real numbers with the finite complement topology. Consider the product and box topologies on $(\mathbb{R}_f)^\omega$.

   Let $g(t) = (\sin t, \sin t, \sin t, \ldots)$ be a function from $\mathbb{R}_f$ to $(\mathbb{R}_f)^\omega$. For which topologies is g continuous?

   Let $h(t) = (t, t, t, \ldots)$ be a function from $\mathbb{R}_f$ to $(\mathbb{R}_f)^\omega$. For which topologies is h continuous?

6. (15 points) Let X be a compact Hausdorff space. Prove that if X has no isolated points, then X is uncountable.

7. (10 points) Prove that the space $\mathbb{R}_l$ is normal.

8. (15 points) Let X be a path connected space. Let a and b be points in X. Prove that $\pi_1(X, a)$ is isomorphic to $\pi_1(X, b)$.

9. (10 points) Prove that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^3$.