MATH 108 – REVIEW TOPIC 10

Quadratic Equations

Introduction:

Any equation that can be expressed in the form $ax^2 + bx + c = 0,\ a \neq 0$ is called a quadratic equation.

Illustration:

$$\begin{array}{rl} 2x^2 + x - 6 = 0 & \text{quadratic in } x \\ -16t^2 + 80t = 0 & \text{quadratic in } t. \end{array}$$

The values that satisfy a quadratic (or any polynomial equation) are called roots.

I. Finding Roots of a Quadratic Equation

There are 3 primary methods for finding roots to a quadratic. Here are examples and comments on each.

A. Factoring

Consider the equation $2x^2 + x - 6 = 0$. When expressed as a polynomial, its roots are not easily apparent. Notice what happens if we rewrite this expression in factored form:

$$2x^2 + x - 6 = 0 \Rightarrow (2x - 3)(x + 2) = 0$$

The roots now become clear: $x = \dfrac{3}{2}$ or $x = -2$. When solving a quadratic equation, factored form has a distinct advantage over polynomial form. Any value that turns a factor into 0 will automatically make the overall product into 0 (and is therefore a root).

In "algebreeze" (that strange language used by math instructors), if $ab = 0$, then $a = 0$ or $b = 0$.

Example: Solve $x(x - 4) = 5$.

Warning: You can only make use of factors when their product is 0.

If the problem read $x(x - 4) = 0$, you would have roots of 0 and 4. No such conclusions about roots can be drawn from $x(x - 4) = 5$.

Our only recourse is to remove parentheses and put into $ab = 0$ form.

Solution:

$$\begin{aligned} x(x-4)=5 &\Rightarrow x^2-4x-5=0 \\ &\Rightarrow (x-5)(x+1)=0 \qquad ab=0 \\ &\Rightarrow x=5 \text{ or } x=-1 \end{aligned}$$

Example: Solve $9x^2 = x$.

Solution:

$$\begin{aligned} 9x^2=x &\Rightarrow 9x^2-x=0 \\ &\Rightarrow x(9x-1)=0 \qquad\qquad ab=0 \\ &\Rightarrow x=0 \text{ or } x=\frac{1}{9} \end{aligned}$$

What if you attempted the same problem using the following method?

$$\begin{aligned} 9x^2=x &\Rightarrow 9x=1 \qquad \text{divide by } x \\ &\Rightarrow x=\frac{1}{9} \end{aligned}$$

Every quadratic equation has 2 roots. Dividing by x removes the root $x=0$. However, dividing by a constant does not effect roots.

Example: Solve $-16t^2+80t=0$.

$$\begin{aligned} -16t^2+80t=0 &\Rightarrow t^2-5t=0 \qquad \text{divide by } (-16) \\ &\Rightarrow t(t-5)=0 \\ &\Rightarrow t=0 \text{ or } t=5 \end{aligned}$$

Here's one last example of how factoring finds roots.

Example: $x^3+3x^2-4x-12=0$.
Even though the example is not quadratic, any factorable polynomial can be solved using the same principles.

Solution:

$$\begin{aligned} & x^3+3x^2-4x-12=0 \\ \Rightarrow\quad & x^2(x+3)-4(x+3)=0 \\ \Rightarrow\quad & (x^2-4)(x+3)=0 \\ \Rightarrow\quad & (x-2)(x+2)(x+3)=0 \\ \Rightarrow\quad & x=\pm 2 \text{ or } x=-3 \end{aligned}$$

Grouping
Review Topic 4

Exercise 1: Solve for x.

a) $x(2x+1) = 15$

b) $12x^2 + 60x + 75 = 0$

c) $\dfrac{5x}{x-2} + \dfrac{3}{x} + 2 = \dfrac{-6}{x^2 - 2x}$ Hint: Find LCD and clear fractions.

d) $x^3 - 9x = 0$

e) $2x^3 - 5x^2 - 18x + 45 = 0$

Answers

B. Quadratic Formula

Another method for finding roots to a quadratic equation is the quadratic formula.

For $ax^2 + bx + c = 0$, $a \neq 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example: Solve $x^2 - 6x - 4 = 0$.

Solution: With $a = 1,\ b = -6$ and $c = -4$,

$$\sqrt{52} = \sqrt{4 \cdot 13} = 2\sqrt{13}$$

$$x = \frac{6 \pm \sqrt{(-6)^2 - 4(1)(-4)}}{2(1)} = \frac{6 \pm \sqrt{52}}{2} = \frac{6 \pm 2\sqrt{13}}{2}$$
$$= 3 \pm \sqrt{13}$$

Any help you need with simplifying radicals can be found in Review Topic 6.

Comments: A quadratic has real roots when $b^2 - 4ac \geq 0$. The discussion of non-real or complex roots (when $b^2 - 4ac < 0$) will be left for the course.

C. "Taking Roots"

For quadratics with no middle term (when $b = 0$), the simplest approach is to take square roots of both sides of the equation.

Example: Solve $3x^2 - 4 = 0$.

Solution:

$$3x^2 - 4 = 0 \Rightarrow x^2 = \frac{4}{3} \qquad \text{isolate } x^2 \text{ and take roots.}$$
$$\Rightarrow x = \pm\frac{2}{\sqrt{3}} = \pm\frac{2\sqrt{3}}{3}.$$

Common error: Don't forget the negative root.

$$x^2 = 5 \Rightarrow x = \sqrt{5} \text{ or } -\sqrt{5}.$$

Example: Solve $(x-2)^2 = 17$.

You could expand the binomial, put into quadratic form and solve. Instead, as in the previous example, square root both sides of the equation.

Solution:

$$(x-2)^2 = 17 \Rightarrow x - 2 = \pm\sqrt{17}$$
$$\Rightarrow x = 2 \pm \sqrt{17}$$

It will help if you keep this example in mind when looking at Section III of this topic, completing the square.

Exercise 2: Find all real solutions.

a) $2x^2 - 10 = 0$

b) $(x+3)^2 = 4$

c) $x^2 - 4x - 3 = 0$

d) $x^2 - 3x + 4 = 0$

e) $\dfrac{x+1}{3x+2} = \dfrac{x-2}{2x-3}$

f) $\dfrac{5}{3}x^2 + 3x + 1 = 0$

g) $k = \dfrac{1}{2}mv^2$ for v (Kinetic energy)

Answers

II. Guidelines for Finding Roots of a Quadratic

You should now be able to solve quadratic equations using any of the three methods shown: factoring, quadratic formula, or taking roots. Here is a summary of what has been covered.

1) For $ax^2+c=0$, isolate x^2 and square root both sides. Don't forget the negative root. Otherwise...

2) Put into the form $ax^2+bx+c=0$. This may require removing parentheses or clearing fractions. Dividing out a constant is helpful but not necessary.

3) Find roots by factoring or the *quadratic formula. If $b^2-4ac<0$, the equation has no real roots.

4) Check solutions, especially if original equation is fractional.

*Don't overuse the quadratic formula. Factoring is an important skill to maintain so use it at every opportunity.

III. Completing the Square

Section Ic) demonstrated how quadratics in the form $(x \pm (\quad))^2 = k$ are solved.

Illustration: $(x-2)^2 = 17 \Rightarrow x-2 = \pm\sqrt{17} \Rightarrow x = 2 \pm \sqrt{17}$.

How is this related to completing the square? By expanding $(x-2)^2$ and setting equal to 0, $(x-2)^2 = 17 \Rightarrow x^2-4x-13=0$. This would seem to indicate that any quadratic can be changed into $(x \pm (\quad))^2 = k$ form (and then solved). Such a process is called completing the square.

A. Perfect Square Trinomials

Completing the square requires a thorough understanding of how trinomials of the form $a^2 \pm 2ab + b^2$ always factor into $(a \pm b)^2$ (Review Topic 4).

Illustration:

$$x^2+6x+9 = \overbrace{x^2+2(3)x+3^2}^{a^2+2ab+b^2} = \overbrace{(x+3)^2}^{(a+b)^2}$$
$$x^2-10x+25 = x^2-2(5)x+5^2 = (x-5)^2$$
$$x^2-5x+\frac{25}{4} = x^2-2\left(\frac{5}{2}\right)x+\left(\frac{5}{2}\right)^2 = \left(x-\frac{5}{2}\right)^2$$

Trinomials such as these are referred to as Perfect Square Trinomials (PST).

Exercise 3: Find the term needed to make a PST, then express in factored form.

$$x^2-5x+\underline{\ ?\ } = x^2-2\left(\frac{5}{2}\right)x+\underline{\ ?\ } = x^2-2\left(\frac{5}{2}\right)x+\underline{\left(\frac{5}{2}\right)^2} = \left(x-\frac{5}{2}\right)^2$$

a) $x^2-14x+\underline{\ ?\ }$ b) $a^2+9a+\underline{\ ?\ }$ c) $x^2-\frac{1}{2}x+\underline{\ ?\ }$ Answers

B. Solving Quadratics by Completing the Square

Example: Solve $x^2-10x-4=0$.

Solution.

$x^2-10x=4$ — Move the constant so it won't interfere with completing the square

$\Rightarrow x^2-2(5)x+\underline{5^2} = 4+\underline{5^2}$ — Comp. Square, maintain equality by adding to both sides

$\Rightarrow (x-5)^2 = 29$

$\Rightarrow x-5 = \pm\sqrt{29}$ — Factor and take roots

$\Rightarrow x = 5\pm\sqrt{29}$

Example: Solve $2x^2 + 7x - 15 = 0$

Solution.

$$x^2 + \frac{7}{2}x = \frac{15}{2} \qquad \text{Divide out coef. of } x^2$$

$$\Rightarrow x^2 + 2\left(\frac{7}{4}\right)x + \left(\frac{7}{4}\right)^2 = \frac{15}{2} + \left(\frac{7}{4}\right)^2$$

$$\Rightarrow \left(x + \frac{7}{4}\right)^2 = \frac{169}{16}$$

$$\Rightarrow x + \frac{7}{4} = \pm\frac{13}{4}$$

$$\Rightarrow x = -\frac{7}{4} \pm \frac{13}{4} = \frac{3}{2} \text{ or } -5$$

Why are these the roots of $2x^2 + 7x - 15 = 0$?

Check:

$$\text{if } x = \frac{3}{2}, \quad 2\left(\frac{3}{2}\right)^2 + 7\left(\frac{3}{2}\right) - 15 = \frac{9}{2} + \frac{21}{2} - \frac{30}{2} = 0$$

$$\text{if } x = -5, \quad 2(-5)^2 + 7(-5) - 15 = 50 - 35 - 15 = 0.$$

Final Comment: Completing the square may not be your preferred method for solving quadratics. However, the process is important to learn. You will need to complete squares when working with equations of circles and parabolas.

Exercise 4: Solve by completing the square.

a) $x^2 + 8x - 6 = 0$

b) $2x^2 - 11x = 0$

c) $5 + 4x - x^2 = 0$

Answers

Solve for x.

a) $x(2x+1) = 15$

b) $12x^2 + 60x + 75 = 0$

c) $\dfrac{5x}{x-2} + \dfrac{3}{x} + 2 = \dfrac{-6}{x^2-2x}$

Hint: Find LCD and clear fractions.

(d) $x^3 - 9x = 0$

(e) $2x^3 - 5x^2 - 18x + 45 = 0$

Answers:

a) $2x^2 + x - 15 = 0 \Rightarrow (2x-5)(x+3) = 0$

$$\Rightarrow x = \frac{5}{2} \text{ or } x = -3$$

b) $12x^2 + 60x + 75 = 0 \Rightarrow 4x^2 + 20x + 25 = 0$

$$\Rightarrow (2x+5)^2 = 0$$

$$\Rightarrow x = -\frac{5}{2}$$

Since $(2x+5)$ is a repeated factor, $-\dfrac{5}{2}$ is a repeated or double root.

c) Multiply by $x(x-2)$ to clear fractions.

$$5x^2 + 3(x-2) + 2x(x-2) = 0$$

$$\Rightarrow 7x^2 - x = 0 \Rightarrow x(7x-1) = 0$$

Thus $x = 0$ or $x = \dfrac{1}{7}$ appear to be roots. Since division by 0 is not permissible, the only root is $\dfrac{1}{7}$. Always check roots when variables appear in any denominator.

d) $x(x^2-9) = 0 \Rightarrow x(x-3)(x+3) = 0$

$$\Rightarrow x = 0 \quad \text{or} \quad \pm 3$$

e)
$$2x^3 - 5x^2 - 19x + 45 = 0$$

$$\Rightarrow x^2(2x-5) - 9(2x-5) = 0$$

$$\Rightarrow (x^2-9)(2x-5) = 0$$

$$\Rightarrow x = \pm 3 \quad \text{or} \quad \frac{5}{2}$$

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Find all real solutions.

a) $2x^2 - 10 = 0$

b) $(x+3)^2 = 4$

c) $x^2 - 4x - 3 = 0$

d) $x^2 - 3x + 4 = 0$

e) $\dfrac{x+1}{3x+2} = \dfrac{x-2}{2x-3}$

f) $\dfrac{5}{3}x^2 + 3x + 1 = 0$

g) $k = \dfrac{1}{2}mu^2$ for v (Kinetic energy)

Answers:

a) $x^2 = 5 \Rightarrow x = \sqrt{5}$ or $-\sqrt{5}$

b) $$x + 3 = \pm 2 \Rightarrow x = -3 \pm 2$$
$$\Rightarrow x = -1 \text{ or } x = -5$$

c) $x = \dfrac{4 \pm \sqrt{27}}{2} = \dfrac{1}{2}(4 \pm 3\sqrt{3})$

d) $x = \dfrac{3 \pm \sqrt{-7}}{2}$; Since $b^2 - 4ac < 0$, this equation has no real roots.

e) Solving as a proportion,

$$(2x-3)(x+1) = (3x+2)(x-2)$$
$$\Rightarrow x^2 - 3x - 1 = 0$$
$$\Rightarrow x = \frac{3 \pm \sqrt{13}}{2}$$

f) After clearing fractions, $5x^2 + 9x + 3 = 0$.

$$x = \frac{-9 \pm \sqrt{21}}{2} = \frac{1}{2}(-9 \pm \sqrt{21})$$

g) $$k = \frac{1}{2}mv^2 \Rightarrow 2k = mv^2 \Rightarrow v^2 = \frac{2k}{m}$$
$$v = \sqrt{\frac{2k}{m}}$$

Since we solved for velocity, disregard the negative root.

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Find the term needed to make a PST, then express in factored form.

a) $x^2 - 14x + \underline{\ ?\ }$ b) $a^2 + 9a + \underline{\ ?\ }$ c) $x^2 - \frac{1}{2}x + \underline{\ ?\ }$

Answers:

a) $x^2 - 2(7)x + 7^2 = (x - 7)^2$

b) $a^2 + 2\left(\frac{9}{2}\right)a + \left(\frac{9}{2}\right)^2 = \left(a + \frac{9}{2}\right)^2$

c) $x^2 - 2\left(\frac{1}{4}\right)x + \left(\frac{1}{4}\right)^2 = \left(x - \frac{1}{4}\right)^2$

Return to Review Topic

Solve by completing the square.

a) $x^2 + 8x - 6 = 0$

b) $2x^2 - 11x = 0$

c) $5 + 4x - x^2 = 0$

Answers:

a)
$$\begin{aligned} (x+4)^2 &= 6 + 4^2 \\ \Rightarrow \quad x + 4 &= \pm\sqrt{22} \\ \Rightarrow \quad x &= -4 \pm \sqrt{22} \end{aligned}$$

b)
$$\begin{aligned} x^2 - \frac{11}{2}x &= 0 \\ \Rightarrow x^2 - 2\left(\frac{11}{4}\right)x + \left(\frac{11}{4}\right)^2 &= \left(\frac{11}{4}\right)^2 \\ \Rightarrow \quad \left(x - \frac{11}{4}\right)^2 &= \left(\frac{11}{4}\right)^2 \\ \Rightarrow \quad x - \frac{11}{4} &= \pm\frac{11}{4} \\ \Rightarrow \quad x &= \frac{11}{4} \pm \frac{11}{4} = 0 \text{ or } \frac{11}{2} \end{aligned}$$

c)
$$\begin{aligned} x^2 - 4x &= 5 \\ \Rightarrow (x-2)^2 &= 9 \\ \Rightarrow \quad x &= 2 \pm 3 = 5 \text{ or } -1 \end{aligned}$$

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