MATH 108 – REVIEW TOPIC 11

Irrational Equations

Introduction

Any equation where the variable is inside a radical is called an irrational equation (numbers inside radicals like $\sqrt{2}$ or $\sqrt[3]{4}$ are irrational numbers). When solving an irrational equation, the key step will be removing the radical.

Example: Solve $3\sqrt{x} - 4 = 5$.

Let's compare this equation to a quadratic of similar form:

$3x^2 - 4 = 5$	$3\sqrt{x} - 4 = 5$
$3x^2 = 9$	$3\sqrt{x} = 9$
$x^2 = 3$	$\sqrt{x} = x^{1/2} = 3 \qquad \sqrt[n]{x^m} = x^{m/n}$
$x = \pm\sqrt{3}$	$x = 3^2 = 9$

In both problems we disregard the exponent and isolate the variable. Exponents are then removed using reciprocal powers, i.e. $\left[(\quad)^{m/n}\right]^{n/m} = (\quad)$.

Example: Solve $3x^{2/3} = 12$.

Solution: $3x^{2/3} = 12 \Rightarrow x^{2/3} = 4$

$$\Rightarrow (x^{2/3})^{3/2} = 4^{3/2}$$

$$\Rightarrow x = (4^{1/2})^3 = (\pm 2)^3 = \pm 8$$

Exercise 1: Solve $\sqrt[3]{2x-5} = 3$. Answer

Warning: Mistakes in radical equations are often the result of not distinguishing between powers of sums versus products.

Illustration:

$$\begin{array}{rll} \text{Product}: & (ab)^m = a^m b^m, & (2\sqrt{x})^2 = 4x \\ \text{Sum}: & (a+b)^m \neq a^m + b^m, & (2+\sqrt{x})^2 = 4 + 4\sqrt{x} + x \end{array}$$

Because of these differences, a problem like $3 + \sqrt{3x+1} = x$ can easily be missed. Let's discuss this problem in detail.

Example: Solve $3 + \sqrt{3x+1} = x$.

One approach would be to square each side of the equation immediately, expecting to remove the radical.

$$\begin{aligned} & (3 + \sqrt{3x+1})^2 = x^2 \\ \Rightarrow \quad & 3^2 + 2(3)\sqrt{3x+1} + (\sqrt{3x+1})^2 = x^2 \qquad (a+b)^2 = a^2 + 2ab + b^2 \\ \Rightarrow \quad & 9 + 6\sqrt{3x+1} + 3x + 1 = x^2. \end{aligned}$$

Comment: Up to this point the algebra is correct, but the radical is still present. Let's start over ...

$$\left.\begin{aligned} 3 + \sqrt{3x+1} = x &\Rightarrow \sqrt{3x+1} = x - 3 \\ &\Rightarrow 3x + 1 = (x-3)^2 \\ &\Rightarrow 3x + 1 = x^2 - 6x + 9 \end{aligned}\right\} \text{See the benefits of isolating the radical}$$

$$\begin{aligned} &\Rightarrow 0 = x^2 - 9x + 8 \\ &\Rightarrow x = 8,\ x = 1 \end{aligned}$$

Check:

$$\begin{array}{ll} \text{if } x = 8, & 3 + \sqrt{25} = 8 \\ \text{if } x = 1, & 3 + \sqrt{4} \neq 1. \end{array}$$

Therefore, the only solution is $x = 8$.

> Raising expressions to powers can lead to additional (extraneous) solutions. You must check to be certain which values are true solutions.

PRACTICE PROBLEMS for Topic 11

Solve the equations.

11.1. $x^{2/3} = 16$

11.2. $x^{-1/2} = 4$

11.3. $2\sqrt{2x-9} - 4 = 0$

11.4. $(2x^2+1)^{1/5} = 2$

11.5. $\sqrt{7-x} = x - 5$

11.6. $x + \sqrt{5x+19} = -1$

11.7. $\sqrt{2\sqrt{x+1}} = \sqrt{3x-5}$

Answers

ANSWERS to PRACTICE PROBLEMS (Topic 11)

11.1. $x = 16^{3/2} = (\pm 4)^3 = \pm 64$

Return to Problem

11.2. $x = 4^{-2} = \dfrac{1}{16}$

Return to Problem

11.3. $\sqrt{2x-9} = 2 \Rightarrow 2x - 9 = 4 \Rightarrow x = \dfrac{13}{2}$

Return to Problem

11.4. $(2x^2 + 1) = 32 \Rightarrow x^2 = \dfrac{31}{2} \Rightarrow x = \pm\dfrac{\sqrt{62}}{2}$

Return to Problem

11.5. $7 - x = x^2 - 10x + 25 \Rightarrow x^2 - 9x + 18 = 0 \Rightarrow x = 6$ or $x = 3$.
$x = 3$ is extraneous, $x = 6$ is the only solution.

Return to Problem

11.6.
$$\begin{aligned} & \sqrt{5x+19} = -1 - x \\ \Rightarrow \quad & 5x + 19 = (-1-x)^2 \\ \Rightarrow \quad & 5x + 19 = 1 + 2x + x^2 \\ \Rightarrow \quad & 0 = x^2 - 3x - 18 \\ \Rightarrow \quad & x = 6 \ \text{ or } \ x = -3 \end{aligned}$$

Check:

if $x = 6$, $\quad 6 + \sqrt{49} \neq -1;\ x = 6$ is extraneous,

if $x = -3$, $\quad -3 + \sqrt{4} = -1;\ x = -3$ is the only root.

Return to Problem

11.7.

$$
\left.\begin{aligned}
& 2\sqrt{x+1} = 3x-5 \\
\Rightarrow\quad & 4(x+1) = 9x^2 - 30x + 25 \qquad (ab)^2 = a^2b^2, \quad (a+b)^2 = a^2+2ab+b^2 \\
\Rightarrow\quad & 0 = 9x^2 - 34x + 21 \\
\Rightarrow\quad & 0 = (9x-7)(x-3) \\
\Rightarrow\quad & x = 7/9 \quad \text{or} \quad x = 3
\end{aligned}\right\}
$$

Quadratic formula leads to $x = \dfrac{34 \pm \sqrt{400}}{18}$ and same result.

only $x = 3$ checks.

Return to Problem

Beginning of Topic 108 Skills Assessment

Solve $\sqrt[3]{2x-5} = 3$.

Answer:

$$\begin{aligned} (2x-5)^{1/3} &= 3 \\ [(2x-5)^{1/3}]^3 &= 3^3 \\ 2x-5 &= 27 \\ x &= 16 \end{aligned}$$

We rewrote the radical to emphasize the use of reciprocal powers.

Return to Review Topic