MATH 108 – REVIEW TOPIC 2

Exponents

I. Counting Factors and Laws of Exponents

What is the purpose of an exponent? The expression b^n indicates that b(called the base) is a factor n times.

Illustration:

x^4 means x is a factor 4 times or $x \cdot x \cdot x \cdot x$

$(x-2)^{10}$ means $(x-2)$ is a factor 10 times.

To "simplify", express the total number of factors with a single exponent.

Example: **<u>How factor is counted</u>**

a) $x^4 \cdot x^3 = \overbrace{x \cdot x \cdot x \cdot x}^{x^4} \cdot \overbrace{x \cdot x \cdot x}^{x^3} = x^7$ — x is a factor $(4+3)$ times

b) $\dfrac{x^{25}}{x^{10}} = \left(\overbrace{\dfrac{x^{10}}{x^{10}}}^{1}\right) x^{15} = x^{15}$ — x is a factor $(25-10)$ times

c) $(x^3)^4 = x^3 \cdot x^3 \cdot x^3 \cdot x^3 = x^{12}$ — x^3 is a factor 4 times so x is a factor (3×4) times

d) $(xy)^3 = xy \cdot xy \cdot xy = x^3y^3$ — xy is a factor 3 times so x AND y are factors 3 times

Based on these examples, we now state the Laws of Exponents.

$$x^m \cdot x^n = x^{m+n} \qquad (xy)^n = x^n y^n$$

$$\frac{x^m}{x^n} = x^{m-n} \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

$$(x^m)^n = x^{m \times n}$$

Rather than just memorize, think of these laws as a quick way to count factors. Once that happens, the laws become **logical** and errors are eliminated.

Exercise 1. Express in simplest form:

a) $x^4(x^{10})$

b) $(x^4)^{10}$

c) $\dfrac{x^4}{x^{10}}$

d) $\dfrac{10^6}{5^8}$ Hint: $10^6 = (2 \cdot 5)^6$

(e) $2x^3(-x^2)^6$

(f) $\dfrac{6x^8}{(4x^3y^2)^2}$

(g) $-3x^5(2x)(4x^2) + (3x^2)^4$

(h) $-6x^7 + x^5$

Answers

II. Negative Exponents

If exponents and laws of exponents count factors for positive exponents, what does a negative or zero exponent count? This is a hard question to answer logically so forgive us for just giving definitions.

Def: $(\ \ )^0 = 1$ for all $(\ \ )$

Def: $(\ \ )^{-m} = \dfrac{1}{(\ \ )^m}$; $\dfrac{1}{(\ \ )^{-m}} = \dfrac{1}{\dfrac{1}{(\ \ )^m}} = (\ \ )^m$

We are using $(\ \ )$ instead of a variable.

Illustration:

a) $4^0 = 1$; $(2+x)^0 = 1$ $\qquad (\ \ )^0 = 1$

b) $(5)^{-2} = \left(\dfrac{1}{5^2}\right) = \dfrac{1}{25}$; $4x^{-3} = 4\left(\dfrac{1}{x^3}\right) = \dfrac{4}{x^3}$ $\qquad (\ \ )^{-m} = \dfrac{1}{(\ \ )^m}$

c) $\dfrac{-2}{3x^{-4}} = -\dfrac{2}{3}\left(\dfrac{1}{x^{-4}}\right) = -\dfrac{2}{3}(x^4) = -\dfrac{2x^4}{3}$ $\qquad \dfrac{1}{(\ \ )^{-m}} = (\ \ )^m$

Keep the following in mind when simplifying expressions with negative exponents:

1) Exponent laws do not depend on the type of exponent.

$$(x^{-2})^{-4} = x^8 \text{ just like } (x^3)^5 = x^{15}$$

2) In a simplified answer all exponents are positive.

$$x^{-4} \cdot x^2 = x^{-2} = \frac{1}{x^2}$$

Examples: Express in simplest form.

a) $\dfrac{4a^{-2}}{3^{-1}b^4}$ b) $(x^{-2}y^4)(x^{-3}y^{-4})$ c) $(-2a^4b^{-1})^{-3}$

Solutions.

a) $\dfrac{4a^{-2}}{3^{-1}b^4} = \dfrac{4 \cdot 3}{a^2b^4} = \dfrac{12}{a^2b^4}$ $\qquad \dfrac{1}{3^{-1}} = 3,\ a^{-2} = \dfrac{1}{a^2}$

b)
$$\begin{aligned}(x^{-2}y^4)(x^{-3}y^{-4}) &= x^{-5}y^0\\ &= \frac{1}{x^5} \cdot 1\\ &= \frac{1}{x^5}\end{aligned}$$

c)
$$\begin{aligned}(-2a^4b^{-1})^{-3} &= (-2)^{-3}a^{-12}b^3\\ &= \frac{1}{(-2)^3} \cdot \frac{1}{a^{12}} \cdot b^3\\ &= -\frac{b^3}{8a^{12}}\end{aligned}$$

Applying exponent laws **before** manipulating to change negative exponents is the simplest approach.

Exercise 2. Express in simplest form.

a) $(8a^3b^{-2})\left(-\dfrac{1}{2}a^4b\right)$

b) $(4x^{-3}y^2)^{-2}$

c) $-2x(3x^{-1}y)^3$

d) $\dfrac{-3m^{-4}n^2}{9m^{-1}n^{-2}}$

e) $\dfrac{(2x^{-3}y)^4}{-4x^3y^5}$

Answers

III. Rational Exponents

Rational (meaning fractional) exponents obey the same laws of exponents.

Illustration:

$$x^{1/2} \cdot x^{1/3} = x^{1/2+1/3} = x^{5/6} \qquad x^m \cdot x^n = x^{m+n}$$

$$(x^{1/2})^4 = x^{1/2(4)} = x^2 \qquad (x^m)^n = x^{m \times n}$$

$$\frac{x}{x^{1/3}} = x^{1-(1/3)} = x^{2/3} \qquad \frac{x^m}{x^n} = x^{m-n}$$

Exercise 3. Simplify.

a) $x^{-1/3} \cdot x^{1/2}$

b) $(a^{-2}b^6)^{-1/3}$

c) $\left(\dfrac{x^{-4}y}{x^2}\right)^{-3/2}$

d) $\dfrac{(x^6y^3)^{-2/3}}{(x^{-2}y^4)^{1/2}}$

Answers

IV. Evaluating Logs

No study of exponents is complete without some mention of logarithms. The expression $\log_a b = x$ in words is read as "log of b to the base $a = x$". What does this mean?

Answer: $\log_a b = x$ means $a^x = b$.

Using blanks, $\log_a b = (\quad)$ is equivalent to $a^{(\quad)} = b$.

Illustration:

$\log_2 8 = ?$ means $2^{\,?} = 8$. Thus $\log_2 8 = 3$ since $2^3 = 8$.

$\log_5 \dfrac{1}{25} = ?$ means $5^{\,?} = \dfrac{1}{25}$. Thus $\log_5 \dfrac{1}{25} = -2$ since $5^{-2} = \dfrac{1}{25}$.

Exercise 4. Evaluate the following.

a) $4^3 =$ ______; $\log_4 64 =$ ______

b) $2^5 =$ ______; $\log_2 32 =$ ______

c) $4^{-2} =$ ______; $\log_4 \frac{1}{16} =$ ______

d) $\log_2 16 =$ ______

e) $\log_{1/3} 9 =$ ______

f) $\log_2 \frac{1}{8} =$ ______

g) $\log_{10} 1000 =$ ______

h) $\log_{10} .01 =$ ______

Answers

Express in simplest form:

a) $x^4(x^{10})$

b) $(x^4)^{10}$

c) $\dfrac{x^4}{x^{10}}$

d) $\dfrac{10^6}{5^8}$ Hint: $10^6 = (2 \cdot 5)^6$

(e) $2x^3(-x^2)^6$

(f) $\dfrac{6x^8}{(4x^3y^2)^2}$

(g) $-3x^5(2x)(4x^2) + (3x^2)^4$

(h) $-6x^7 + x^5$

Answers:

a) x^{14}

b) x^{40}

c) $\left(\dfrac{x^4}{x^4}\right) \cdot \dfrac{1}{x^6} = \dfrac{1}{x^6}$

d) $\dfrac{2^6 \cdot 5^6}{5^8} = \dfrac{2^6}{5^2} = \dfrac{64}{25}$

e) $2x^3(x^{12}) = 2x^{15}$

f) $\dfrac{6x^8}{16x^6y^4} = \dfrac{3x^2}{8y^4}$

g) $-24x^8 + 81x^8 = 57x^8$

h) Already in simplest form. Sorry for trying to trick you.

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Express in simplest form.

a) $(8a^3b^{-2})\left(-\dfrac{1}{2}a^4b\right)$

b) $(4x^{-3}y^2)^{-2}$

c) $-2x(3x^{-1}y)^3$

d) $\dfrac{-3m^{-4}n^2}{9m^{-1}n^{-2}}$

e) $\dfrac{(2x^{-3}y)^4}{-4x^3y^5}$

Answers:

a) $-4a^7b^{-1} = -\dfrac{4a^7}{b}$

b) $4^{-2}x^6y^{-4} = \dfrac{x^6}{16y^4}$

c) $-2x(27x^{-3}y^3) = -54x^{-2}y^3 = -\dfrac{54y^3}{x^2}$

d) $-\dfrac{1}{3}m^{-3}n^4 = -\dfrac{n^4}{3m^3}$

e) $\dfrac{16x^{-12}y^4}{-4x^3y^5} = -4x^{-15}y^{-1} = -\dfrac{4}{x^{15}y}$

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Simplify.

a) $x^{-1/3} \cdot x^{1/2}$

b) $(a^{-2}b^6)^{-1/3}$

c) $\left(\dfrac{x^{-4}y}{x^2}\right)^{-3/2}$

d) $\dfrac{(x^6y^3)^{-2/3}}{(x^{-2}y^4)^{1/2}}$

Answers:

a) $x^{-1/3+1/2} = x^{1/6}$

b) $a^{2/3}b^{-2} = \dfrac{a^{2/3}}{b^2}$

c) $\left(\dfrac{y}{x^6}\right)^{-3/2} = \left(\dfrac{x^6}{y}\right)^{3/2} = \dfrac{x^9}{y^{3/2}}$

d) $\dfrac{x^{-4}y^{-2}}{x^{-1}y^2} = \dfrac{1}{x^3y^4}$

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Evaluate the following.

a) $4^3 =$ ______; $\log_4 64 =$ ______

b) $2^5 =$ ______; $\log_2 32 =$ ______

c) $4^{-2} =$ ______; $\log_4 \frac{1}{16} =$ ______

d) $\log_2 16 =$ ______

e) $\log_{1/3} 9 =$ ______

f) $\log_2 \frac{1}{8} =$ ______

g) $\log_{10} 1000 =$ ______

h) $\log_{10} .01 =$ ______

Answers:

a) 64; 3

b) 32; 5

c) $\frac{1}{16}$; -2

d) 4

e) -2; $\left(\frac{1}{3}\right)^{-2} = 9$

f) -3; $2^{-3} = \frac{1}{8}$

g) 3

h) -2

NOTE: Base 10 logs are called common logs and have their own special notation. $\log_{10} a$ is written as $\log a$. Thus $\log 1000 = 3$ and $\log .01 = -2$.

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