MATH 108 – REVIEW TOPIC 4

Factoring

Introduction. Factoring is a process for rewriting a certain type of expression as a product. Actually, factoring can be thought of as the reverse of multiplication. Most forms of multiplication (distributing, squaring a binomial, multiplying two binomials, etc.) can be related to some type of factoring.

Let us illustrate.

The product of $2x$ and (x^2-3y) is $2x^2-6xy$. That is, $2x(x^2-3y) = 2x^3-6xy$.

Notice we started with factors and ended with an expression. Can this process be reversed? Suppose we require $2x^3 - 6xy$ to be expressed as a product of factors. From above, we have

$$2x^3 - 6xy = 2x(x^2 - 3xy).$$

This type of factoring is possible when each term has a "shared" or common factor.

There are other types of factoring. Consider $(x-3)(2x+5) = 2x^2 - x - 15$. That means when factored, $2x^2 - x - 15 = (x-3)(2x+5)$.

In general there are four main types of factoring: Greatest Common Factoring (GCF), Trinomial Factoring, Difference of Two Squares, and Sum or Difference of Two Cubes. Here are several examples of each type and some brief comments.

1a. Greatest Common Factor

Here the idea is to see if all the terms have anything in common. If so, factor it (or "pull it out") from each term.

Example: Factor $x^2y - 3xy^4$.
Since the two terms have xy in common, we "pull out" xy:

$$x^2y - 3xy^4 = xy(\overbrace{\underline{\ ?\ } - \underline{\ ?\ }}^{\text{what remains}}) = xy(\underbrace{x - 3y^3}_{\substack{\text{nothing common}\\ \text{remains within}}}).$$

Example:
$$\begin{aligned} 2(a+b)^2 + 6(a+b)^3 &= 2(a+b)^2[\underline{\ ?\ } + \underline{\ ?\ }] \\ &= 2(a+b)^2[1 + 3(a+b)] \\ &= 2(a+b)^2(1 + 3a + 3b). \end{aligned}$$

Example: $a^{1/3} - 4a^{4/3} = a^{1/3}(\underline{\ ?\ } - 4\underline{\ ?\ }) = a^{1/3}(1 - 4a)$

Notice from these last two examples that even when the common factor is a "quantity" or contains fractional exponents, the process is the same. Namely, remove as much common "stuff" as possible and leave the rest.

1b. Grouping

A special type of common factoring is referred to as "grouping". This usually involves four terms with two steps of common factoring.

Example:

$$\begin{aligned} x^3 - 2x^2 + 4x - 8 &= (x^3 - 2x^2) + (4x - 8) && \text{Group in pairs} \\ &= x^2(x-2) + 4(x-2) \\ &= (x^2 + 4)(x - 2) \end{aligned}$$

Try a different grouping (possibly $x^3 + 4x - 2x^2 - 8$) and see what result you get.

IMPORTANT: As in all factoring problems, multiplying the factors should yield the original expression. This is a good way to check if your factoring is correct.

Exercise 1: Factor each:

a) $12x^3y - 18x^2y^2 - 24x^5$ Answer

b) $x^{2/3} + 2x^{5/3}$ Answer

c) $3(x+2)^{-1/2} + x(x+2)^{1/2}$ Answer

d) $2x^3 - 6x^2 - 5x + 15$ Answer

2a. General Trinomials

By this time you've done "hundreds" of problems requiring you to multiply two binomials (usually called FOIL). Trinomial factoring requires similar thinking, only in reverse. In particular, you must select and arrange the terms in each factor so that the outer and inner products have a sum equal to the middle term.

Here are some examples.

$$\begin{aligned}
x^2 - 7xy + 10y^2 &= (x - _\, y)\ (x - _\, y) = (x - 2y)\ (x - 5y)\\
x^2y^2 - 3xy - 10 &= (x - \ _)\ (xy + \ _) = (xy - 5)\ (xy + 2)\\
x^4 - 9x^2 - 10 &= (x^2 - \ _)\ (x^2 + \ _) = (x^2 - 10)\ (x^2 + 1)\\
4x^2 + 16x + 15 &= \big((\ )x + \ _\big)\big((\ )x + \ _\big) = (2x + 5)\ (2x + 3)\\
4x^2 - 17x - 15 &= \big((\ )x + \ _\big)\big((\ )x - \ _\big) = (4x + 3)\ (x - 5)
\end{aligned}$$

Note: A test for factorability of a trinomial is shown in the answer to Exercise 2.

2b. Perfect Square Trinomials (PST)

There's a special trinomial that results when any binomial is squared. Recall $(a \pm b)^2 = a^2 \pm 2ab + b^2$. Therefore any trinomial that matches this distinct pattern factors into the square of a binomial.

Illustration:

$$\begin{aligned}
x^2 - 6x + 9 = (x-3)^2 &\quad \text{because} \quad -2(x)(3) = -6x\\
25x^2 + 30xy + 9y^2 = (5x + 2y)^2 &\quad \text{because} \quad 2(5x)(3y) = 30xy
\end{aligned}$$

If the middle term does not satisfy this "$2ab$" condition, the trinomial still may factor as a general trinomial.

Example: $4x^2 - 29x + 25$

This is not a PST. The first and third terms have the form a^2 and b^2, but the middle term is not of the form $-2(2x)(5) = -20x$. However it still factors.

Ans: $4x^2 - 29x + 25 = (4x - 25)(x - 1)$

Exercise 2: Factor each:

a) $x^2 - 72x + 20$

b) $x^2 - 21x^6y + 20y^2$

c) $x^2 - 9x - 20$

d) $x^2 - 10x + 25$

(e) $2x^2 - 11x + 15$

(f) $2x^4 - 7x^2 - 15$

(g) $4x^2 + 12x + 9$

(h) $4x^2 - 13x + 9$

(i) $x^{2n} - 3x^n - 4$

Answers

3. Difference of Two Squares

This is probably the easiest type of factoring to recognize. It follows a distinct pattern.

$$a^2 - b^2 = (a - b)(a + b)$$

Example: $9x^2 - 25 = (3x)^2 - (5)^2 = (3x - 5)(3x + 5)$
$4x^6 - 49 = (2x^3)^2 - (7)^2 = (2x^3 - 7)(2x^3 + 7)$

WARNING!! Watch out for expressions of the form $a^2 + b^2$.

Common error: $x^2 + 16 = (x + 4)^2$ is not true.

Squaring a binomial yields a middle term of the form $2ab$.

$$(x + 4)^2 = x^2 + \underline{8x} + 16$$

That means $x^2 + 16$ (or the sum of any two squares) is nonfactorable.

Exercise 3: Factor each:

a) $9 - 4x^2$

b) $a^2b^4 - 16$

c) $x^2 + 25$

(d) $x^6 - 100$

(e) $x^2 - 5$

Answers

4. Sum or Difference of 2 Cubes

Let's begin by finding the following products.

a) $(x-3)(x^2+3x+9)$ b) $(x^2+2)(x^4-2x^2+4)$

Did you get a) x^3-27 and b) x^6+8?

This means x^3-27 factors into $(x-3)(x^2+3x+9)$ and x^6+8 into $(x^2+2)(x^4-2x^2+4)$. Notice that x^3-27 and x^6+8 represent a difference or sum of 2 cubes: $x^3-27=(x)^3-(3)^3$ and $x^6+8=(x^2)^3+(2)^3$. Maybe the secret to cube factoring will come from analyzing these products?

How does the product of a binomial and trinomial yield only 2 terms? Let's examine our first product:

$$3x^2 \qquad -3x^2$$
$$(x-3)(x^2+3x+9)$$
$$-9x \qquad +9x$$

Because two pairs of "like" terms sum to 0, the only terms remaining in the final answer come from

$$-27$$
$$(x-3)(x^2+3x+9)=x^3-27$$
$$x^3$$

The key to cube factoring is found in these products.

Exercise 4: Below are products of binomials and trinomials. Each product results in a sum or difference of two cubes. Fill in the blanks.

a) $(x-2)(x^2+\underline{\ ?\ }+4)=x^3-8$

b) $(2x+1)(4x^2-\underline{\ ?\ }+\underline{\ ?\ })=8x^3+1$

c) $(a^3-b^3)=(a-b)(\underline{\ ?\ }+\underline{\ ?\ }+\underline{\ ?\ })$

d) $(x^3+125y^3)=(\underline{\ ?\ }+\underline{\ ?\ })(x^2-\underline{\ ?\ }+\underline{\ ?\ })$ Answers

You should now be ready (if not eager) to factor cubes.

Example: Factor $x^3 + 1$.

$$x^3 + 1 = (x)^3 + (1)^3$$ "Sum of cubes"

$-x^2$ x^2

$$= (x + 1)(x^2 + \underline{\ ?\ } + 1) = (x + 1)(x^2 - 1x + 1)$$

$-1x$ $+1x$

only $x^3 + 1$ remains

Example: Factor $8x^3 - 27$.

$$8x^3 - 27 = (2x)^3 - (3)^3$$ "Difference of cubes"

$12x^2$

$$= (2x - 3)(4x^2 + \underline{\ ?\ } + 9) = (2x - 3)(4x^2 + 6x + 9)$$

$-12x^2$

NOTEWORTHY:

* The trinomial can't be factored. Verify by applying the test for factorability ($b^2 - 4ac$).

** The common error mentioned before (with squares) still applies.

$x^2 + 16 \neq (x + 4)^2$ Expanding $(x + 4)^2$ yields 3 terms

$x^3 + 27 \neq (x + 3)^3$ Expanding $(x + 3)^3$ yields 4 terms

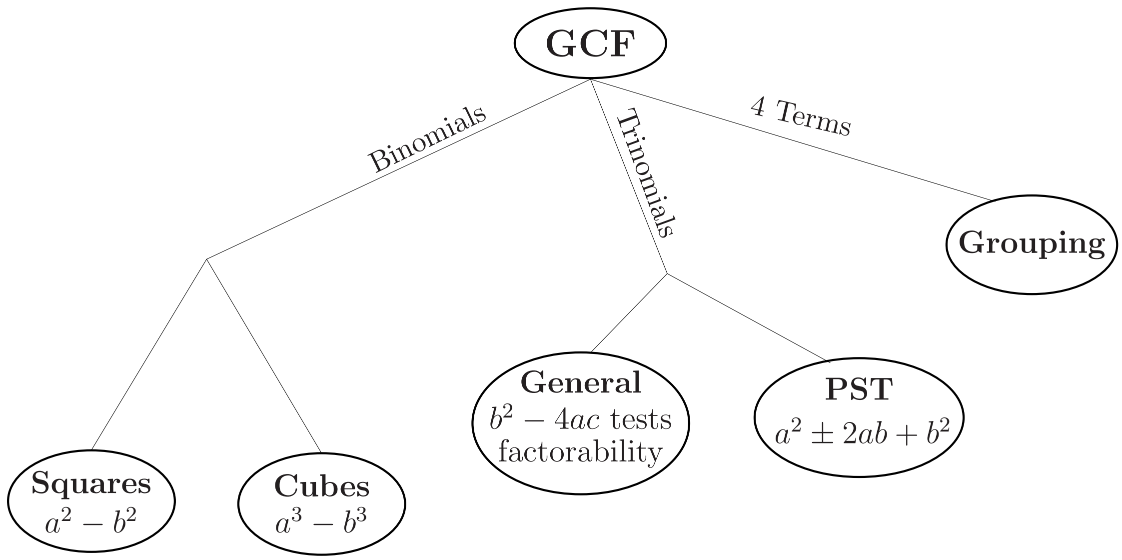
*** To avoid factoring mistakes, check by multiplying.

Exercise 5: Factor:

a) $a^3b^3 - 8$ b) $x^6 + 64$ Answers

II. Guidelines for Factoring

The real challenge in factoring is twofold: recognizing which type to use and mixed factorizations (more than one type involved). Start by looking for the common factor, then proceed according to the number of terms.



Here are four examples. Notice how we follow this strategy.

Ex.
$$\begin{aligned} 50x - 2x^3 &= 2x(25 - x^2) && \text{GCF} \\ &= 2x(5-x)(5+x) && \text{Dif of 2} \end{aligned}$$

Ex.
$$\begin{aligned} 4y^3 - 10y^2 - 6y &= 2y(2y^2 - 5y - 3) \Rightarrow b^2 - 4ac = 49 && \text{GCF} \\ &= 2y(2y+1)(y-3) && \text{Gen. Tri.} \end{aligned}$$

Ex.
$$\begin{aligned} x^6 - 7x^3 - 8 &= (x^3+1)(x^3-8) && \text{Gen. Tri.} \\ &= (x+1)(x^2-x+1)(x-2)(x^2+2x+4) && \text{Cubes} \end{aligned}$$

Ex.
$$\begin{aligned} x^2 + 4x + 4 - 9y^2 &= (x^2+4x+4) - 9y^2 && \text{3-1 Grouping} \\ &= (x+2)^2 - 9y^2 && \text{PST} \\ &= (x+2)^2 - (3y)^2 \\ &= [(x+2) - 3y][(x+2) + 3y] && \text{Dif of 2 sq.} \end{aligned}$$

It is important that you master factoring. In the review topics that follow, note how often factoring is used and for what purposes.

PRACTICE PROBLEMS for Topic 4

4.1. Factor completely.

a) $2x^2(x+1)^3 + 5x(x+1)^2$ Answer

b) $-2x^3 - 6x^2 + 8x$ Answer

c) $2x^2 + x - 21$ Answer

d) $4x^2 - 5x - 6$ Answer

e) $4x^2 - 5x + 6$ Answer

(d, e) Test for factorability using $b^2 - 4ac$.

f) $x^4 - 10x + 9$ Answer

g) $x^6 - 1$ Answer

h) $x^3 + 3x^2 - 4x - 12$ Answer

i) $x^6 - 7x^3 - 8$ Answer

ANSWERS to PRACTICE PROBLEMS (Topic 4–Factoring)

4.1. a) $x(x+1)^2(2x+5)$ Return to Problem

b) $-2x(x^2+3x-4) = -2x(x+4)(x-1)$ Return to Problem

c) $(2x+7)(x-3)$ Middle term: $-6x+7x = x$ Return to Problem

d) $b^2-4ac = 121 = 11^2$ guarantees factors.
$(4x+3)(x-2)$ Return to Problem

e) $b^2-4ac = -71$; Nonfactorable or prime. Return to Problem

f) $(x^2-9)(x^2-1) = (x-3)(x+3)(x-1)(x+1)$ Return to Problem

g) As Difference of Cubes:

$$\begin{aligned}(x^2)^3-(1)^3 &= (x^2-1)(x^4+x^2+1)\\ &= (x-1)(x+1)(x^4+x^2+1)\end{aligned}$$

As Difference of Squares:

$$\begin{aligned}(x^3)^2-(1)^2 &= (x^3-1)(x^3+1)\\ &= (x-1)(x^2+x+1)(x+1)(x^2-x+1)\end{aligned}$$

Comment: It is unusual that 2 correct methods "appear" to give different results. This means $x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$. Try to find a way to factor x^4+x^2+1, we dare you.

Return to Problem

h) $x^2(x+3)-4(x+3) = (x^2-4)(x+3) = (x-2)(x+2)(x+3)$

Return to Problem

i) $(x^3-8)(x^3+1) = (x-2)(x^2+2x+4)(x+1)(x^2-x+1)$

Return to Problem

Beginning of Topic 108 Skills Assessment

Factor each:

a) $12x^3y - 18x^2y^2 - 24x^5$

b) $x^{2/3} + 2x^{5/3}$

c) $3(x+2)^{-1/2} + x(x+2)^{1/2}$

d) $2x^3 - 6x^2 - 5x + 15$

Answers:

a) $6x^2(2xy - 3y^2 - 4x^3)$

b) $x^{2/3}(1+2x)$

c) $(x+2)^{-1/2}[3 + x(x+2)] = (x+2)^{-1/2}(x^2+2x+3)$

d) $2x^2(x-3) - 5(x-3) = (2x^2-5)(x-3)$

Return to Review Topic

Factor each:

a) $x^2 - 72x + 20$

b) $x^2 - 21x^6y + 20y^2$

c) $x^2 - 9x - 20$

d) $x^2 - 10x + 25$

(e) $2x^2 - 11x + 15$

(f) $2x^4 - 7x^2 - 15$

(g) $4x^2 + 12x + 9$

(h) $4x^2 - 13x + 9$

(i) $x^{2n} - 3x^n - 4$

Answers:

a) $(x - 10)(x - 2)$

b) $(x - 20y)(x - y)$

c) Prime

d) $(x - 5)^2$

e) $(2x - 5)(x - 3)$

f) $(2x^2 + 3)(x^2 - 5)$

g) $(2x + 3)^2$

h) $(4x - 9)(x - 1)$

i) $(x^n - 4)(x^n + 1)$

Maybe your trinomial factoring is already good. Here is how it can get better.

How to test a trinomial for factorability:

Given any trinomial in the form $ax^2 + bx + c$, evaluate $b^2 - 4ac$.

i) If $b^2 - 4ac = 0$, the trinomial is a PST $\Rightarrow$

ii) If $b^2 - 4ac$ is a perfect square (1, 4, 9, ...) then the trinomial factors as a general trinomial.

iii) All other values indicate the trinomial is non factorable.

Return to Review Topic

Factor each:

a) $9 - 4x^2$

b) $a^2b^4 - 16$

c) $x^2 + 25$

(d) $x^6 - 100$

(e) $x^2 - 5$

Answers:

a) $(3 - 2x)(3 + 2x)$

b) $(ab^2 - 4)(ab^2 + 4)$

c) Prime

d) $(x^3 - 10)(x^3 + 10)$

e) Prime

Comment: $(x-\sqrt{5})(x+\sqrt{5}) = x^2-(\sqrt{5})^2 = x^2-5$? Shouldn't that mean $x^2 - 5$ is factorable? Yes and No. If we factor over the integers, $x^2 - 5$ is not factorable. If we factor over the reals (which include roots) then $x^2 - 5$ factors into $(x - \sqrt{5})(x + \sqrt{5})$.

Illustration: Factor $x - 10$ over the reals.

Ans: $(\sqrt{x} - \sqrt{10})(\sqrt{x} + \sqrt{10})$.

Return to Review Topic

Below are products of binomials and trinomials. Each product results in a sum or difference of two cubes. Fill in the blanks.

a) $(x-2)(x^2 + \underline{\ ?\ } + 4) = x^3 - 8$

b) $(2x+1)(4x^2 + \underline{\ ?\ } + \underline{\ ?\ }) = 8x^3 - 1$

c) $(a^3 - b^3) = (a-b)(\underline{\ ?\ } + \underline{\ ?\ } + \underline{\ ?\ })$

d) $(x^3 + 125y^3) = (\underline{\ ?\ } + \underline{\ ?\ })(x^2 + \underline{\ ?\ } + \underline{\ ?\ })$

Answers:

a) $(x-2)(x^2 + \underline{2x} + 4)x^3 - 8$

b) $(2x+1)(4x^2 - \underline{2x} + \underline{1}) = 8x^3 + 1$

c) $(a^3 - b^3) = (a-b)(\underline{a^2} + \underline{ab} + \underline{b^2})$

d) $x^3 + 125y^3 = (\underline{x} + \underline{5y})(x^2 - \underline{5xy} + \underline{25y^2})$

In each of the problems, the sign between the terms of the binomial factor matches the sign between the two cube terms in the result.

Return to Review Topic

Factor:

a) $a^3b^3 - 8$

b) $x^6 + 64$

Answers:

a)
$$\begin{aligned} a^3b^3 - 8 &= (ab)^3 - (2)^3 \\ &= (ab-2)(a^2b^2 + 2ab + 4) \end{aligned}$$

b)
$$\begin{aligned} x^6 + 64 &= (x^2)^3 + (4)^3 \\ &= (x^2+4)(x^4 + \underline{\quad} + 16) \\ &= (x^2+4)(x^4 - 4x^2 + 16) \end{aligned}$$

There are other ways to find the trinomial factor. There is a formula,

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2).$$

Long division also works. In Exercise b), $x^2 + 4\overline{\smash{)}\,x^6 + 64}$ yields the quotient, and other factor, $x^4 - 4x^2 + 16$. Isn't this to be expected? You've known about the factor/division relationship since your earliest days learning arithmetic facts.

Return to Review Topic