MATH 109 – TOPIC 10
TRIGONOMETRIC EQUATIONS

Solving a trig equation is a lot like baking a cake (my favorite is German chocolate). Start with a big bowl. Then mix in the proper ingredients including exact values and identities. Finally, stir in some algebra and pop in the oven (or in this case your brain). Heat for several days. For an added treat, cover with fat-free graphs. Are you hungry yet?

I. Basic Equations

Let's start with the essentials.

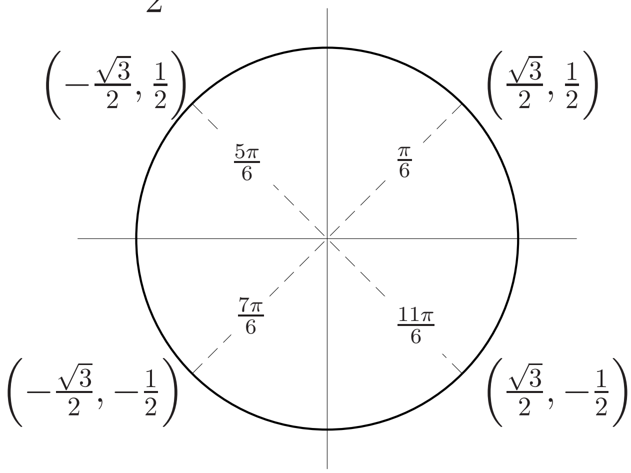
Example 10.1. Find all θ on the interval $[0, 2\pi)$ such that $\sin\theta = -\dfrac{1}{2}$. Here are 3 methods that will enable you to solve and interpret the results.

A. Unit Circle

If you recall from Topic 5, ordered pairs on the unit circle represent the values of sine and cosine. Specifically, if θ is in standard position,

$$\mathbf{x = \cos\boldsymbol{\theta} \quad and \quad y = \sin\boldsymbol{\theta}}.$$

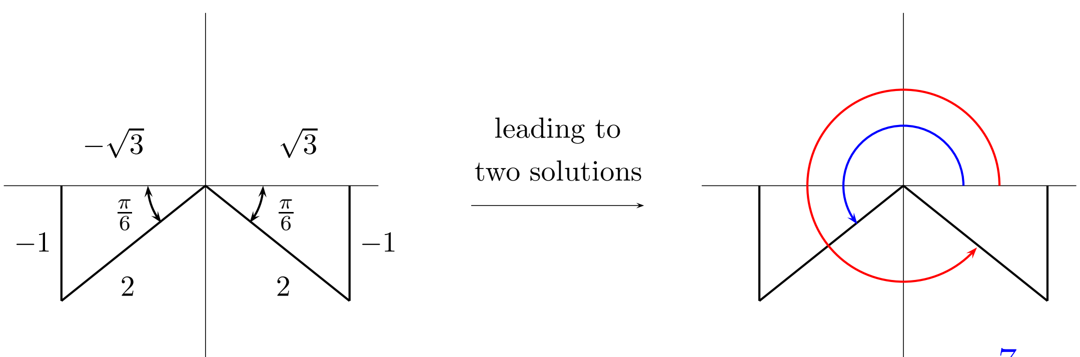
To solve $\sin\theta = -\frac{1}{2}$, simply select all angles whose y coordinate is $-\frac{1}{2}$.



Solution: $\sin\theta = -\frac{1}{2}$ for $\theta = \frac{7\pi}{6}$ and $\frac{11\pi}{6}$.

B. Right Triangles

This method relies on exact values and reference angles. Begin by finding the quadrants where θ terminates. In our example, with $\sin\theta < 0$, θ terminates in III or IV. Using the appropriate right triangle with legs 1, $\sqrt{3}$ and hypotenuse 2, we get:



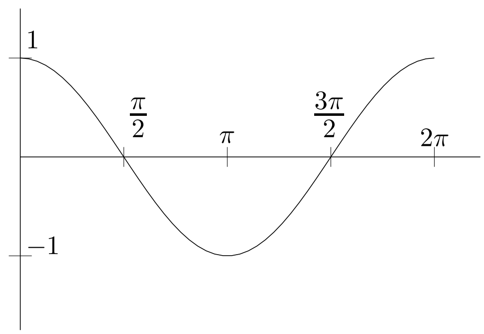
$\frac{\pi}{6}$ is the reference angle

$$\theta_1 = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

$$\theta_2 = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

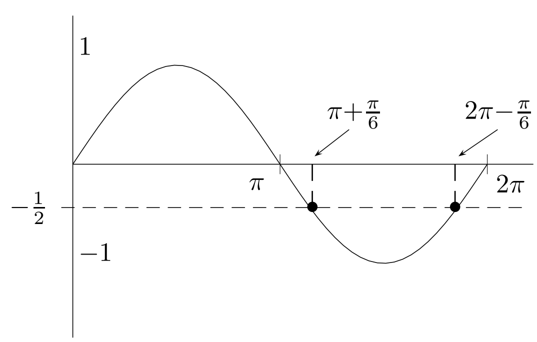
C. Functions and Graphs

This method works best when working with exact values associated with quadrantal angles $(\ldots\ -\pi, -\dfrac{\pi}{2}, 0, \dfrac{\pi}{2}, \pi,\ \ldots)$. Suppose you wish to solve $\cos x = 0$ on $[0, 2\pi)$? Start with the graph of $y = \cos x$:



Now just read off all the angles whose output is 0. Since $\cos\dfrac{\pi}{2} = \cos\dfrac{3\pi}{2} = 0$, the two solutions on $[0, 2\pi)$ for $\cos x = 0$ are $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$.

Even our original example $\left(\sin\theta = -\dfrac{1}{2}\right)$, can be explained using a graph. Just look for the intersection of $y = \sin x$ with the horizontal line $y = -\dfrac{1}{2}$.



I recommend methods B and C. Most of trig can be explained by triangles, functions or graphs.

PRACTICE PROBLEMS:

10.1. Solve on $[0, 2\pi)$.

$$\left.\begin{array}{ll} \text{a)} & \cos x = 0 \\ \text{b)} & \sin x = -1 \\ \text{c)} & \csc x = 1 \end{array}\right\} \text{Try using graphs.}$$

d) $\tan x = 1$

e) $\tan x = -1$

f) $\sin x = -\dfrac{\sqrt{3}}{2}$

g) $\cos x = \dfrac{1}{\sqrt{2}}$

h) $\sec x = 2$

If you use triangles, here is a good aid that shows where functions are positive.

Example: tan is positive in I and III.

sin	all
tan	cos

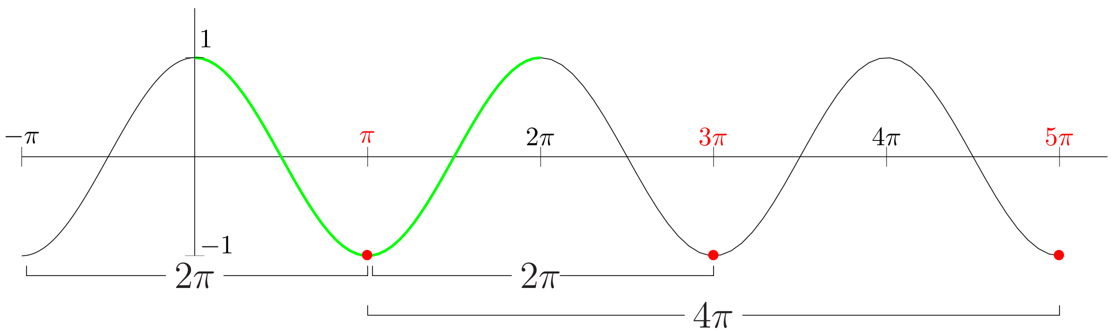
Answers

II. Finding "All Solutions"

Finding solutions on $[0, 2\pi)$ is a good start, but sometimes we need to find **all solutions**. We talked earlier (Topic 7) about periodic behavior of trig functions and their graphs. Let's examine the significance periodicity has on equations.

Example 10.2: Find all solutions to $\cos x = -1$.

Let's look at a cosine graph that has been extended.



On $[0, 2\pi)$, only $\cos\pi = -1$. Thus the only solution is $x = \pi$. But when you consider the extended graph (due to periodicity), $\cos x$ has many more outputs of -1. In fact, $\cos x = -1$ has an infinite number of solutions (all 2π apart). As a solution set we could express this as $\left\{ \ldots \overbrace{\pi - 4\pi}^{-3\pi}, \overbrace{\pi - 2\pi}^{-\pi}, \pi, \overbrace{\pi + 2\pi}^{3\pi}, \overbrace{\pi + 4\pi}^{5\pi} \ldots \right\}$. More likely, this same solution is written

$$x = \pi + 2k\pi \text{ where } k \text{ is an integer.}$$

Try setting $k = \{-2, -1, 0, 1, 2\}$ and you'll see why this works.

Suppose we want to find all solutions to $\cos x = 0$. Once again, it helps to read a graph. From the graph above, $\cos x$ is at 0 when $x = \left\{ \ldots -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \right\}$. Since solutions are all π apart, we can write

$$x = \frac{\pi}{2} + \pi k.$$

Exercise 1: Find all solutions for $\sin x = -\dfrac{1}{2}$.
In part I we found solutions to this equation on $[0, 2\pi)$. Try writing all solutions and then click on answer to check. Answer

To finish parts I and II, here are solutions to several equations, both on $[0, 2\pi)$ and all solutions.

Equation	$[0, 2\pi)$	All Solutions
$\sin x = 0$	$x = 0,\ \pi$	$x = \pi k$
$\sin x = -\dfrac{\sqrt{3}}{2}$	$x = \dfrac{4\pi}{3},\ \dfrac{5\pi}{3}$	$x = \dfrac{4\pi}{3} + 2\pi k$ $x = \dfrac{5\pi}{3} + 2\pi k$
$\csc x = \dfrac{1}{2}$	NO SOLUTION $\csc x = \dfrac{1}{2} \implies \sin x = 2$ But range of $\sin x$ is $[-1, 1]$	
$\cos x = \dfrac{1}{\sqrt{2}}$	$x = \dfrac{\pi}{4},\ \dfrac{7\pi}{4}$	$x = \dfrac{\pi}{4} + 2\pi k$ $x = \dfrac{7\pi}{4} + 2\pi k$
$\tan x = -\sqrt{3}$	$x = \dfrac{2\pi}{3},\ \dfrac{5\pi}{3}$	$x = \dfrac{2\pi}{3} + \pi k$

With "all solutions" the key is not just the period. You still must determine how long it takes for the graph to return to a similar point. Then you'll know why to add on πk, $2\pi k$, or whatever is appropriate.

III. Algebra Really Matters

Even though we are discussing trig equations, algebra is still the key. You simply have to recognize and apply the appropriate algebra. But this won't be so easy since the algebra is in trig form. Here are several examples. Note the similarities in algebra used in both versions. We've left off the final solutions for the trig equations so that you might continue to practice.

Example 10.3.

1) Linear Equations

$$\begin{aligned} x &= 5x + 2 \\ -4x &= 2 \\ x &= -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \sin\theta &= 5\sin\theta + 2 \\ -4\sin\theta &= 2 \\ \sin\theta &= -\frac{1}{2} \\ \theta &= \underline{\ ?\ } \qquad \text{Solve on } [0, 2\pi) \end{aligned}$$

2) Quadratic Equations

a)

$$\begin{aligned} 4x^2 - 3 &= 0 \\ x^2 &= \frac{3}{4} \\ x &= \pm\sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} 4\sin^2\theta - 3 &= 0 \\ \sin^2\theta &= \frac{3}{4} \\ \sin\theta &= \pm\frac{\sqrt{3}}{2} \\ \theta &= \underline{\ ?\ } \end{aligned}$$

b)

$$\begin{aligned} 2x^2 - x - 3 &= 0 \\ (2x-3)(x+1) &= 0 \\ x &= \frac{3}{2}\,,\ x = -1 \end{aligned}$$

$$\begin{aligned} 2\cos^2\theta - \cos\theta - 3 &= 0 \\ (2\cos\theta - 3)(\cos\theta + 1) &= 0 \\ \cos\theta &= \frac{3}{2}\,,\ \cos\theta = -1 \\ \theta &= \underline{\ ?\ } \end{aligned}$$

c)
$$\begin{aligned} x(x-1) &= 2 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2,\ x &= -1 \end{aligned}$$

$$\begin{aligned} \sec\theta(\sec\theta - 1) &= 2 \\ \sec^2\theta - \sec\theta - 2 &= 0 \\ (\sec\theta - 2)(\sec\theta + 1) &= 0 \\ \sec\theta = 2,\ \sec\theta &= -1 \\ \theta &= \underline{\ ?\ } \end{aligned}$$

3) Equations Involving Substitution

$$\begin{aligned} 2y^2 - x - 1 = 0 &\text{ with } x^2 + y^2 = 1 \\ 2(1-x^2) - x - 1 &= 0 \\ -2x^2 - x + 1 &= 0 \\ 2x^2 + x - 1 &= 0 \\ (2x-1)(x+1) &= 0 \\ x = \frac{1}{2},\ x &= -1 \end{aligned}$$

$$\begin{aligned} 2\sin^2\theta - \cos\theta - 1 &= 0 \\ 2(1-\cos^2\theta) - \cos\theta - 1 &= 0 \\ -2\cos^2\theta + \cos\theta - 1 &= 0 \\ 2\cos^2\theta + \cos\theta - 1 &= 0 \\ (2\cos\theta - 1)(\cos\theta + 1) &= 0 \\ \cos\theta = \frac{1}{2},\ \cos\theta &= -1 \\ \theta &= \underline{\ ?\ } \end{aligned}$$

Answers

Is the algebra getting easier to recognize? Now is a good time to see.

Exercise 2. Solve on $[0, 2\pi)$.

a) $2\tan x - \sec x = 0$

b) $2\sin\theta\cos\theta + 2\sin\theta + \cos\theta + 1 = 0$

c) $\sin\theta - \sqrt{3}\cos\theta = 0$

Answers

Beginning of Topic Skills Assessment

1. Linear Equations
   $\sin\theta = -\frac{1}{2} \implies \theta = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

2. Quadratic Equations

   a) $\sin\theta = \dfrac{\sqrt{3}}{2} \implies \theta = \dfrac{\pi}{3}$ or $\dfrac{2\pi}{3}$;
   $\sin\theta = -\dfrac{\sqrt{3}}{2} \implies \theta = \dfrac{4\pi}{3}$ or $\dfrac{5\pi}{3}$

   b) $\cos\theta = \dfrac{3}{2}$ has no solutions. Cosine has range $[-1, 1]$.
   $\cos\theta = -1 \implies \theta = \pi$

   c) $\sec\theta = 2 \implies \cos\theta = \frac{1}{2} \implies \theta = \dfrac{\pi}{3}$ or $\dfrac{5\pi}{3}$
   $\sec\theta = -1 \implies \cos\theta = -1 \implies \theta = \pi$

3. Equations Involving Substitution
   $\theta = \dfrac{\pi}{3}$, π, or $\dfrac{5\pi}{3}$

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ANSWERS to PRACTICE PROBLEMS (Topic 10–Trigonometric Equations)

a) $x = \frac{\pi}{2}, \frac{3\pi}{2}$ $\quad \cos\frac{\pi}{2} = \cos\frac{3\pi}{2} = 0$

b) $x = \frac{3\pi}{2}, \quad \sin\frac{3\pi}{2} = -1$

c) $\csc x = 1 \implies \sin x = 1, \quad x = \frac{\pi}{2}$

d) $x = \frac{\pi}{4}, \frac{5\pi}{4}$ Ref. angle $= \frac{\pi}{4}$; x terminates in I or III.

e) $x = \frac{3\pi}{4}, \frac{7\pi}{4}$

f) $x = \frac{4\pi}{3}, \frac{5\pi}{3}$ Ref. angle $= \frac{\pi}{3}$; x terminates in III or IV.

g) $x = \frac{\pi}{4}, \frac{7\pi}{4}$ Ref. angle $= \frac{\pi}{4}$; x terminates in I or IV.

h) $\sec x = 2 \implies \cos x = \frac{1}{2}, \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$

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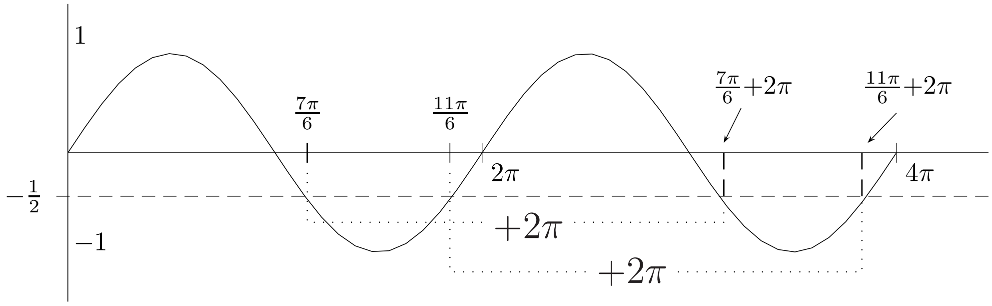
Exercise 1.



Once again, periodic behavior is causing inputs that are 2π apart to have identical outputs. Thus for $\sin x = -\dfrac{1}{2}$, $x = \dfrac{7\pi}{6} + 2\pi k$ or $\dfrac{11\pi}{6} + 2\pi k$.

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Exercise 2.

a) $2\tan x - \sec x = 0$

$\implies 2\left(\dfrac{\sin x}{\cos x}\right) - \dfrac{1}{\cos x} = 0$ Identity substitution

$\implies 2\sin x - 1 = 0$ Mult by $\cos x$

$\sin x = \frac{1}{2} \implies x = \dfrac{\pi}{6}$, or $\dfrac{5\pi}{6}$

b) $2\sin\theta\cos\theta + 2\sin\theta + \cos\theta + 1 = 0$

$$\left.\begin{aligned} &\implies 2\sin\theta(\cos\theta + 1) + 1(\cos\theta + 1) = 0 \\ &\implies (2\sin\theta + 1)(\cos\theta + 1) = 0 \end{aligned}\right\} \text{ Factor by grouping.}$$

$\sin\theta = -\frac{1}{2} \implies \theta = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

$\cos\theta = -1 \implies \theta = \pi$

c) $\sin\theta - \sqrt{3}\cos\theta = 0$

$\implies \sin\theta = \sqrt{3}\cos\theta$

$\implies \dfrac{\sin\theta}{\cos\theta} = \sqrt{3}$

$\implies \tan\theta = \sqrt{3}$

$\theta = \dfrac{\pi}{3}$ or $\dfrac{4\pi}{3}$

Be careful with division. Used improperly, it causes you to "Lose" solutions.

Example: $\cos^2 x - \cos x = 0$ should be solved by factoring (not by dividing by $\cos x$!).

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