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MATH 109 – TOPIC 9

INVERSE TRIGONOMETRIC FUNCTIONS

I. Introduction

When this topic is discussed in algebra, several concepts are involved. Let's examine the three that are most critical.

1. If $f(x)$ is a function with inverse $g(x)$, usually notated by $f^{-1}(x)$, then $f(a) = b \Leftrightarrow g(b) = a$.

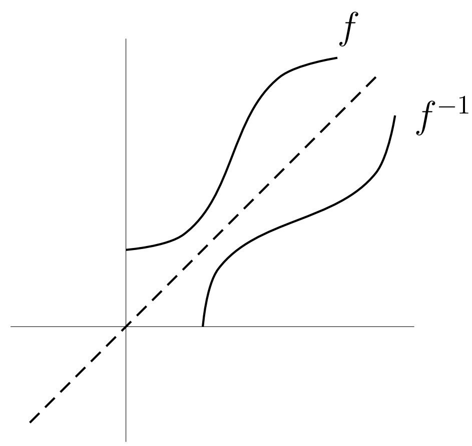
   Translation: $f(3) = 5$ means $f^{-1}(5) = 3$.

2. For any function f to have an inverse, it must be 1–1. That means two inputs cannot have the same output ($f(x_1) \neq f(x_2)$). Functions that are not 1–1 can still have an inverse, but only after the inputs are restricted.

3. Given a function, there are two ways to graph its inverse.

   a) Take (a, b) points from f and plot them as (b, a) points on f^{-1}, or

   b) rotate the graph of f about the line $y = x$ (as demonstrated below).



Exercise 1. Consider $f(x) = x^2$.

a) Explain why f is not a 1–1 function.

b) Show how to restrict the domain of f to make it 1–1 while still maintaining its entire range.

c) Find $f^{-1}(4)$ and $f^{-1}(9)$.

d) Sketch a graph of f^{-1}.

Answers

PRACTICE PROBLEMS for Topic 9 – (Math 109 Inverse Trigonometric Functions)

9.1 Given a table of values for the function f, state the ordered pairs for f^{-1}.

x	$f(x)$
2	5
3	4
-1	-1
0	-3

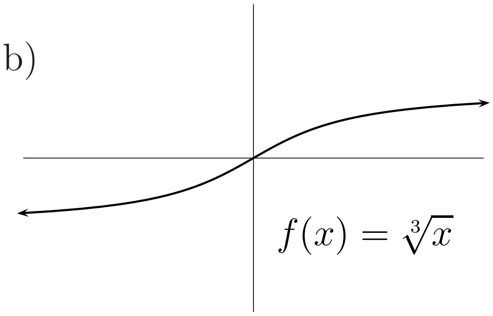
Answers

9.2 Which of the following functions do not exhibit 1–1 behavior?

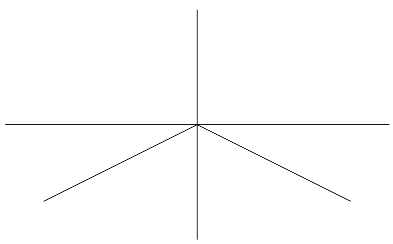
a)

x	$f(x)$
1	4
−1	2
3	2
0	5

b)

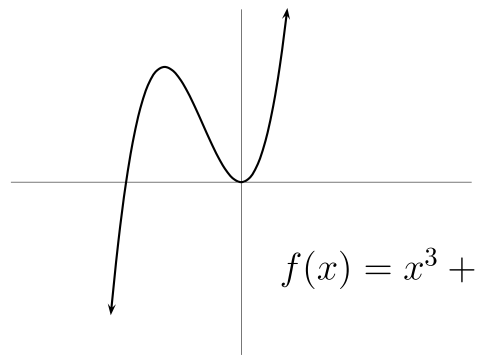


c)



$f(x) = -|x|$

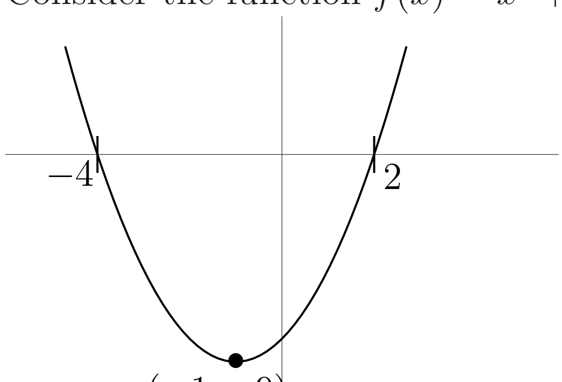
d)



$f(x) = x^3 + 3x^2$

Answers

9.3 Consider the function $f(x) = x^2 + 2x - 8$ whose graph is given by



Indicate the interval that would make f invertible and keep its range $[-9, \infty)$.

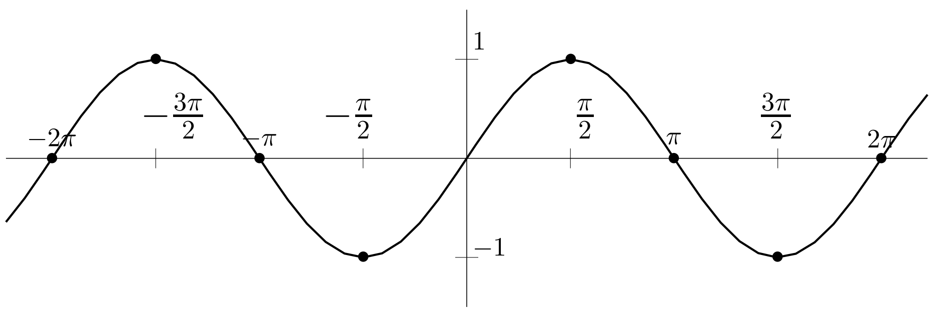
Answer

II. Inverse Trigonometric Functions

We hope the introduction was helpful. Let's begin this discussion by finding the inverse of the sine function.

From Topic 7, we see that $f(x) = \sin x$ has the following graph.

$\sin x, \ -\infty < x < \infty$



Remember that in order for a function $f(x)$ to have an inverse, its graph must be one-to-one. This can be verified by applying the Horizontal Line Test or HTL (a horizontal line intersects the graph at most once). Clearly the graph of $\sin x$ fails this test. So how do we define the inverse of $\sin x$? Easily. We merely restrict the domain of $\sin x$ to the interval $-\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ as shown in Fig. 9.1 below. On this interval, the graph does pass the HLT yet keeps its full range $[-1, 1]$. Thus, $\sin^{-1} x$ or $\arcsin x$ exists.

$\sin x, -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$

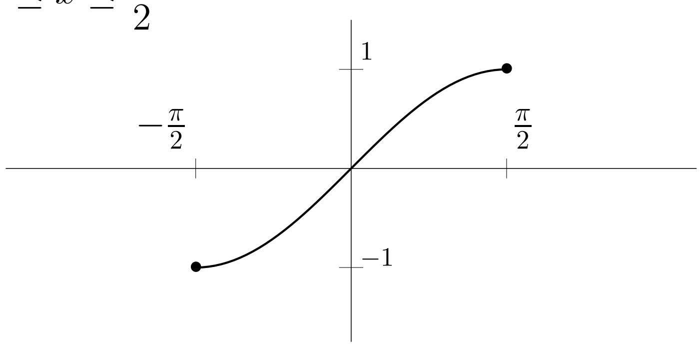
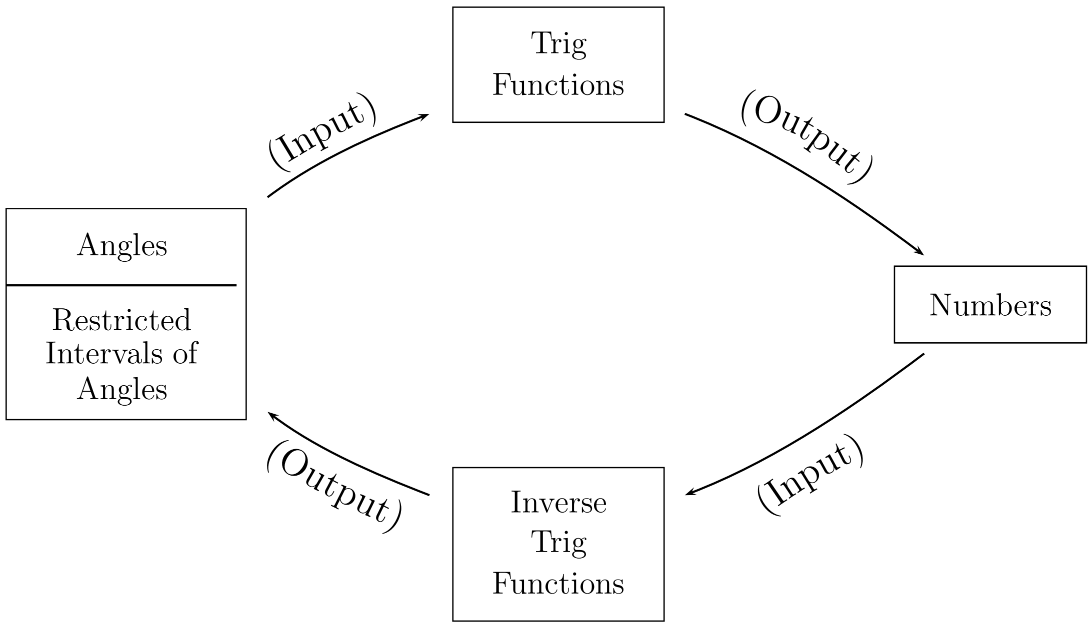


Fig. 9.1.
Note: This is not the graph of $\sin^{-1} x$. It is merely the part of the $\sin x$ function that is used to graph the inverse sine function.

Let's now discuss some of the properties of $\arcsin x$. First, $\sin^{-1} x$ or $\arcsin x$ "reverses" $\sin x$. That is, if $\sin\frac{\pi}{2} = 1$, then $\sin^{-1} 1$ or $\arcsin 1 = \frac{\pi}{2}$. (This is true for inverse functions in general. That is, if $f(a) = b$, then $f^{-1}(b) = a$.) Similarly, since $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$, then $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

Warning! We know $\sin\frac{5\pi}{6} = \frac{1}{2}$, but $\arcsin\frac{1}{2} \neq \frac{5\pi}{6}$!! Why not?! Because $\arcsin\frac{1}{2}$ must be an angle in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In particular, since $\sin\frac{\pi}{6} = \frac{1}{2}$, we conclude $\arcsin\frac{1}{2} = \frac{\pi}{6}$ (not $\frac{5\pi}{6}$). The picture below may help.



As the diagram shows, the sine function has angles as inputs and numbers as outputs, whereas $\arcsin(x)$ has numbers as inputs and **angles** in a restricted interval as outputs. Sometimes this can be helpful in evaluating $\arcsin(x)$. For example, $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$ is the **angle** in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{\sqrt{3}}{2}$. Thus $\arcsin\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$.

PRACTICE PROBLEMS for Topic 9 – (Math 109 Inverse Trigonometric Functions)

9.4 Evaluate the following without using a calculator. (Remember the interval for inverse sin is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$).

a) $\arcsin \frac{1}{\sqrt{2}}$; b) $\sin^{-1} 1$; c) $\arcsin(-1)$;

d) $\arcsin 2$; e) If $\sin \frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, then $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) = ?$

Answers

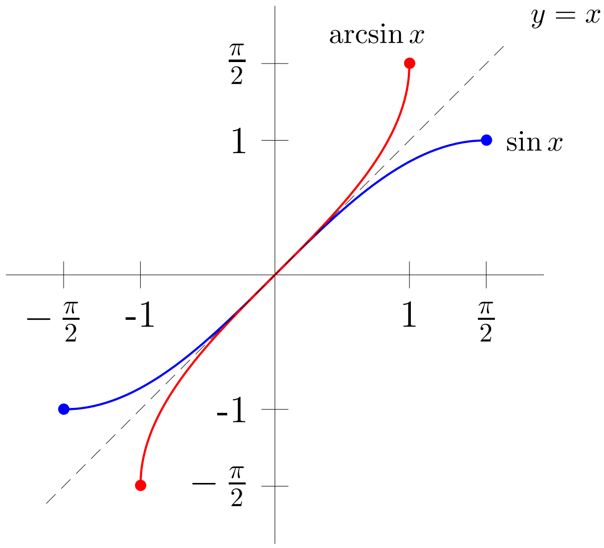
III. Graphs of Inverse Trig Functions

This section discusses the graphs of the inverse trigonometric functions. Using Fig. 9.1, the graph of $\arcsin x$ or $\sin^{-1} x$ can be easily obtained. The domain of $\sin x$ (the set of inputs) is restricted to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and the range (the set of outputs) is $[-1, 1]$. For the inverse function these "flip-flop". Therefore we have the following.

$$\text{For } \arcsin x, \text{ domain} = [-1, 1] \text{ and range} = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

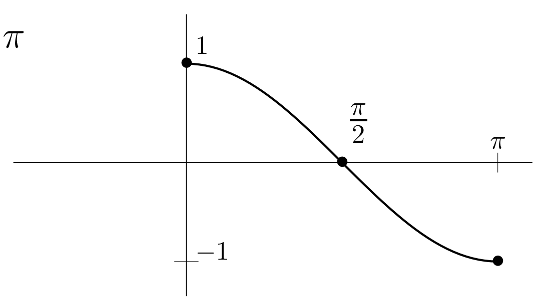
Remember that if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. This means that the graph of f^{-1} can be obtained by rotating the graph of f about the line $y = x$. Using this idea of rotation and/or by plotting some points, we see that $\arcsin x$ has the following graph.

$\sin x,\ -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}$ $\qquad\qquad$ $\arcsin x$ or $\sin^{-1} x$



The derivations of the graphs of $\arccos x$ and $\arctan x$ are similar to that of $\arcsin x$. In both instances, intervals must be restricted. For $\arccos x$, we use the following 1–1 piece.

$\cos x,\ 0 \le x \le \pi$



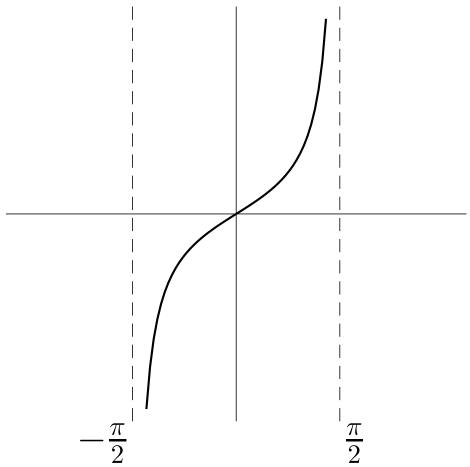
On this restricted domain, $\arccos x$ or $\cos^{-1} x$ exists.

Exercise 2. Find the exact values of $\cos \dfrac{7\pi}{6}$, $\cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ and $\arccos(-1)$.

Answer

To derive $\arctan x$, we must use the 1–1 piece shown below.

$$\tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$$



Once again, our graph passes the HLT and so $\tan^{-1} x$ or $\arctan x$ exists.

PRACTICE PROBLEMS for Topic 9 – Inverse Trigonometric Functions

9.5 Without using a calculator:

a) evaluate $\arccos \dfrac{1}{\sqrt{2}},\ \arccos 0, \arccos\left(-\dfrac{1}{2}\right)$.

b) what is the domain of $\arccos x$?

c) what is the range of $\arccos x$?

d) graph $\arccos x$. Answers

9.6 Without using a calculator:

a) evaluate $\arctan 1, \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$, and $\arctan 0$.

b) find the domain and range of $\arctan x$.

c) graph $\arctan x$. Answers

IV. Composition of Functions

Let's find out what you remember about composition (and then apply it to trig).

Exercise 3. Suppose $f(x) = 3x - 5$, $g(x) = \dfrac{x+5}{3}$, and $h(x) = \sqrt{x+1}$. Find the following:

a) $(f \circ h)(x)$

b) $(f \circ g)(-5)$

c) $(f \circ f)(-5)$

d) $f(g(1))$ Hint: Same as $(f \circ g)(1)$

e) $g(f(4))$

Check your answers and then continue. Answers

Using these results (and what you can infer), state the value of the following compositions. Remember, f and g are inverse functions.

h) $f\left(g\left(-\dfrac{37}{40}\right)\right)$

i) $g(f(-100))$

j) $\sin\left(\arcsin\dfrac{\pi}{4}\right)$

k) $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right)$

Now check your answers and see what conclusions may be drawn. Answers

Summary

Be careful with compositions, and especially those involving inverse trig functions. Oftentimes the cancellation you expect will occur. But just as often, due to the interval restrictions that exist in all trig inverses, the correct result will only come if you work out the composition one step at a time.

PRACTICE PROBLEM for Topic 9 – Composition of Functions

9.7 Find the exact value of each.

a) $\sin\left(\arcsin\dfrac{1}{2}\right)$ d) $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right)$

b) $\sin\left(\arccos\dfrac{1}{2}\right)$ e) $\sin\left(\arccos\dfrac{1}{3}\right)$

c) $\cos^{-1}\left(\cos\dfrac{\pi}{3}\right)$

Answers

Beginning of Topic 109 Study Topics 109 Skills Assessment

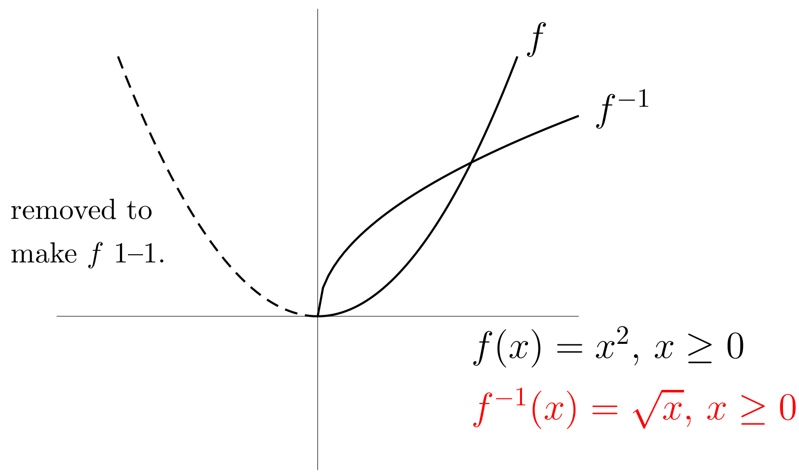
Exercise 1.

a) $f(1) = f(-1)$, $f(2) = f(-2)$, etc. Different inputs cannot have the same output.

b) To remove repeated outputs and still maintain the entire range $[0, \infty)$, select $x \geq 0$. The resulting function is now 1–1.

c) $f(2) = 4 \Rightarrow f^{-1}(4) = 2$
$f(3) = 9 \Rightarrow f^{-1}(9) = 3$

d)



Trig inverses are somewhat more complicated, but understanding the algebra involved in inverse functions is a good beginning.

Return to Review Topic

Exercise 2.

$$\cos\frac{7\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is asking for the angle (on the restricted interval $[0,\pi]$) whose cosine is $-\frac{\sqrt{3}}{2}$. Thus $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$.

$$\arccos(-1) = \cos^{-1}(-1) = \pi$$

Return to Review Topic

Exercise 3. $f(x) = 3x - 5$, $g(x) = \dfrac{x+5}{3}$, $h(x) = \sqrt{x+1}$

a) $(f \circ h)(x) = f(\sqrt{x+1}) = 3\sqrt{x+1} - 5$
$h(x)$ is the input for f.

b) $(f \circ g)(-5) = f(0) = -5$
$g(-5) = 0$ becomes the input for f.

c) $(f \circ f)(-5) = f(-20) = -65$

d) $f(g(1)) = f(2) = 1$

e) $g(f(4)) = f(7) = 4$

In each case, the initial input is the final output. The outer function seems to "undo" the inner (which is typical when algebra inverses are composited).

h) $f\left(g\left(-\dfrac{37}{40}\right)\right) = -\dfrac{37}{40}$

i) $g(f(-100)) = -100$

j) $\sin\left(\arcsin\dfrac{\pi}{4}\right) = \dfrac{\pi}{4}$

k) $\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right) = -\dfrac{\pi}{6}$!!!
What happened here???

The reason these inverses did not undo each other is due to the restricted interval concept discussed in parts II and III. Let's go over this problem step-by-step:

$\sin^{-1}\left(\sin\dfrac{7\pi}{6}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right)$. Now the restriction on the range of $\arcsin x$, $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, forces us to choose $-\dfrac{\pi}{6}$ (instead of returning to $\dfrac{7\pi}{6}$).

Return to Review Topic

IV. ANSWERS to PRACTICE PROBLEMS (Topic 9–Inverse Trigonometric Functions)

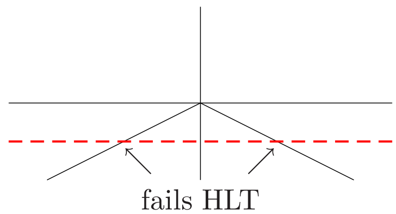
9.1. $f^{-1}(5) = 2$, $f^{-1}(4) = 3$, $f^{-1}(-1) = -1$, $f^{-1}(-3) = 0$.

Return to Problem

9.2. a) Not 1–1. Inputs -1 and 3 both have the same output.

b) 1–1. Passes horizontal line test.

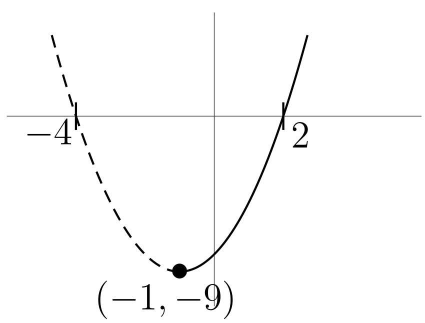
c) Not 1–1.



d) Not 1–1. Fails HLT.

Return to Problem

9.3. This piece is 1–1 and still has range $[-9, \infty]$.



Return to Problem

9.4. a) $\arcsin \dfrac{1}{\sqrt{2}} = \dfrac{\pi}{4}$;

b) $\sin^{-1} 1 = \dfrac{\pi}{2}$;

c) $\arcsin\left(-\dfrac{1}{2}\right) = -\dfrac{\pi}{6}$;

d) $\arcsin 2$ has no solution.
There is no angle whose sine $= 2$.

e) $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) =$ angle in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ whose sine is $\left(-\dfrac{\sqrt{2}}{2}\right)$.
So, $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right) = -\dfrac{\pi}{4}$.

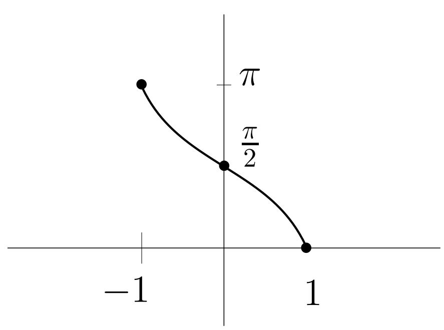
Return to Problem

9.5. a) $\arccos \dfrac{1}{\sqrt{2}} =$ <u>angle</u> <u>in</u> <u>the</u> <u>interval</u> <u>$[0, \pi]$</u> whose cosine is $\dfrac{1}{\sqrt{2}}$.
So, $\arccos \dfrac{1}{\sqrt{2}} = \dfrac{\pi}{4}$. $\arccos 0 = \dfrac{\pi}{2}$; $\arccos\left(-\dfrac{1}{2}\right) = \dfrac{2\pi}{3}$.

b) Domain $= [-1, 1]$

c) Range $= [0, \pi]$

d) Graph of $\arccos x$



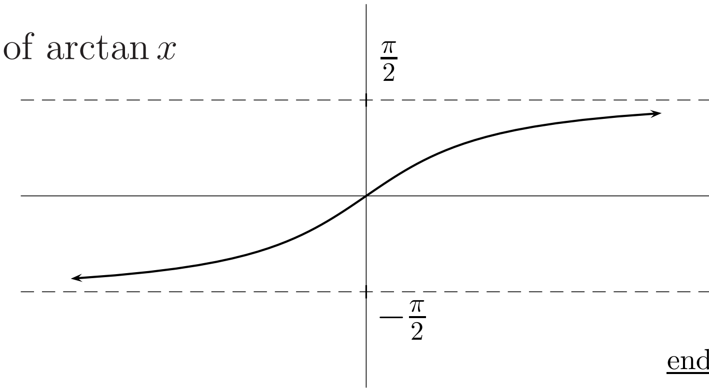
Return to Problem

9.6. a) $\arctan 1 = \underline{\text{angle}}\ \underline{\text{in}}\ \underline{\text{the}}\ \underline{\text{interval}}\ \underline{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)}$ whose tan is 1

$$= \frac{\pi}{4}\,;\ \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}\,; \quad \arctan 0 = 0.$$

b) Domain $= (-\infty, \infty)$; range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

c) Graph of $\arctan x$



<u>end behavior</u>

as $x \to \infty$, $f(x) \to \pi/2$

as $x \to -\infty$, $f(x) \to -\pi/2$

Return to Problem

9.7. a) $\sin\left(\arcsin\frac{1}{2}\right) = \frac{1}{2}$

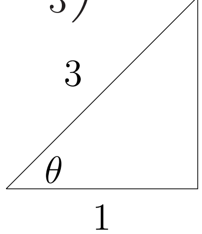
Exactly what you expect. Inverses undo one another.

b) $\sin\left(\arccos\frac{1}{2}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

c) $\cos^{-1}\left(\cos\frac{\pi}{3}\right) = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$

d) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right) = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

e) A ltitle tricky. There is no **exact** angle θ such that $\cos\theta = \frac{1}{3}$, but a picture helps. Thus $\sin\left(\arccos\frac{1}{3}\right) = \sin\theta = \frac{2\sqrt{2}}{3}$.



$\sqrt{3^2 - 1^2} = 2\sqrt{2}$

Return to Problem