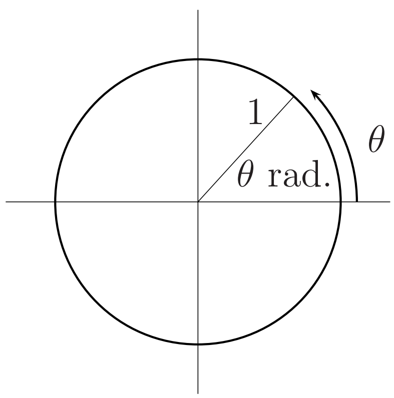
Math 150 – Topic 11

# UNIT CIRCLE DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

In this section we derive the trigonometric functions based on the concept of a unit circle.

Consider a unit circle (radius = 1) centered at the origin. Review Topic 10 studied the formula $s = r\theta$. This relates the length of an arc of a circle with the radius of the circle and the central angle $\theta$ (in radians). For the unit circle, $r = 1$ and so $s = \theta$. That is, the arc length subtended ("marked off") by an angle is equal to the angle (in radians). We have the following picture, Fig. 11.1.




$$\theta = \text{arc length}$$
$$= \text{angle (in radians)}$$

Fig. 11.1. Unit Circle

Let $(x, y)$ be the coordinates of a point $P$ on the unit circle as it moves in a counterclockwise direction from the positive $x$-axis. The point $P$ also determines an angle $\theta$ (in radians) which, as mentioned above, is equal to the length of the arc traveled by $P$. See Fig. 11.2.

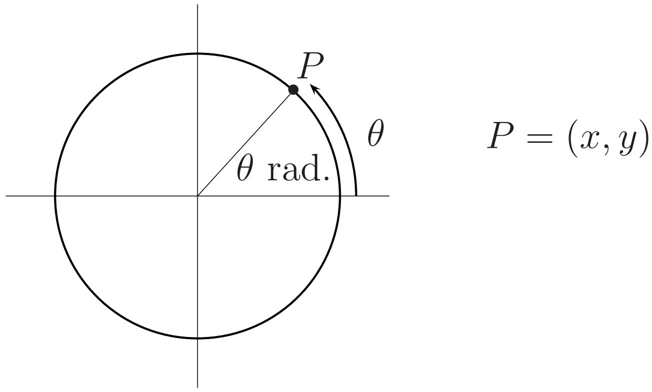



Fig. 11.2

Now, in this same circle, let us draw a right triangle as in Fig. 11.3.

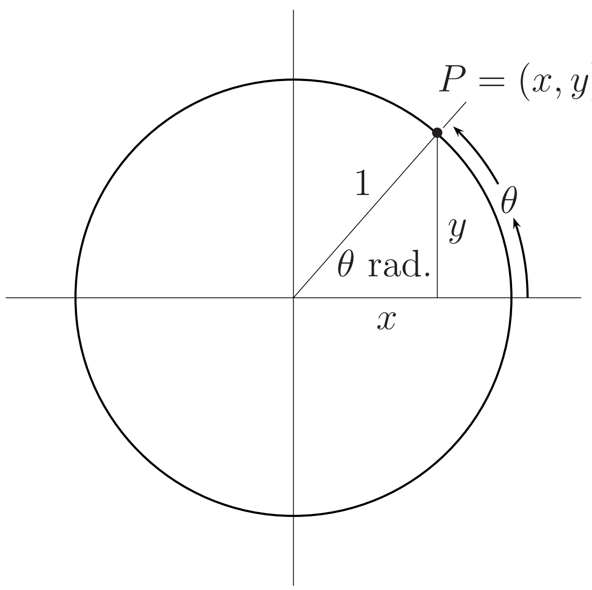



Fig. 11.3

Using the right triangle definitions of the trigonometric functions (Review Topic 9a), we can write

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = y,$$

and

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = x.$$

In other words, we see that $\cos\theta = x$ and $\sin\theta = y$. Once $\sin\theta$ and $\cos\theta$ are determined, the other trigonometric functions are defined in the same manner as in the right triangle approach, Review Topic 9a. This leads to the following alternative definition of the trigonometric functions, the unit circle definition.

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For a unit circle (radius = 1), a central angle measuring $\theta$ radians subtends an arc of length $\theta$. This arc corresponds to a point $P$ with coordinates $(x, y)$. Then,

$$\cos\theta = x \quad \text{and} \quad \sin\theta = y.$$

That is, finding the cosine or sine of $\theta$ is the same as finding the $x$ or $y$ coordinate of $P$.

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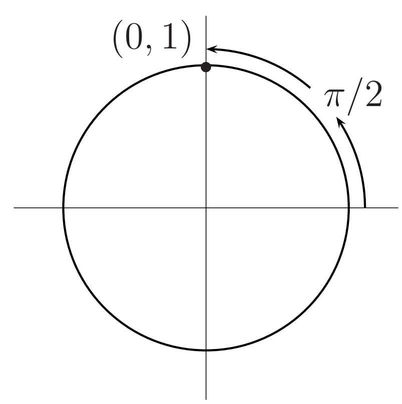
Maybe an example will help.

**Example:** Find $\cos\frac{\pi}{2}$, $\sin\frac{\pi}{2}$ and $\tan\frac{\pi}{2}$.

Solution: $\theta = \dfrac{\pi}{2}$ corresponds to $P = (0, 1)$. Thus,

$$\begin{aligned}\cos\frac{\pi}{2} &= x = 0,\\ \sin\frac{\pi}{2} &= y = 1, \text{ and}\\ \tan\frac{\pi}{2} &= \frac{y}{x} = \text{undefined},\end{aligned}$$

since we cannot divide by 0.




In a similar fashion, $\theta = \pi$ corresponds to $P(-1, 0)$. This means $\cos\pi = x$ coordinate $= -1$ and $\sin\pi = y = 0$. You should now be able to complete the table below.

| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|
| $\sin\theta$ | | | | |
| $\cos\theta$ | | | | |
| $\tan\theta$ | | | | |

Table 11.4

There are other values of $\theta$ for which we know the $x$ and $y$ coordinates. Based on Figures 9b.2 and 9b.5 in Review Topic 9b, we can list the coordinates of $P$ for $\theta = \frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{3}$ as indicated below.

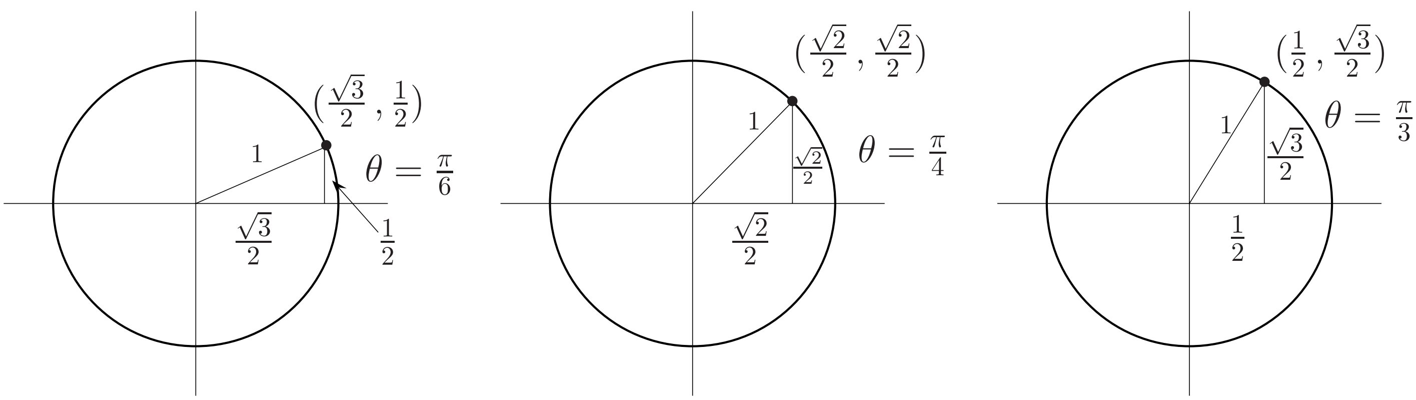



Fig. 11.5

This means $\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$, $\sin\frac{\pi}{4} = y = \frac{\sqrt{2}}{2}$, etc, which matches the values obtained in the right angle definition.

As $P$ moves from one quadrant to the next around the unit circle, the $x$ and $y$ coordinates will repeat themselves or have the opposite values of $x$ and $y$ coordinates in previous quadrants. For example, $\frac{\pi}{6}$ in the first quadrant matches the point $P\left(\dfrac{\sqrt{3}}{2},\dfrac{1}{2}\right)$. Evidently, $\dfrac{5\pi}{6}$, $\dfrac{7\pi}{6}$, and $\dfrac{11\pi}{6}$ all correspond to points whose coordinates are related to $P\left(\dfrac{\sqrt{3}}{2},\dfrac{1}{2}\right)$ as indicated on the unit circle below.

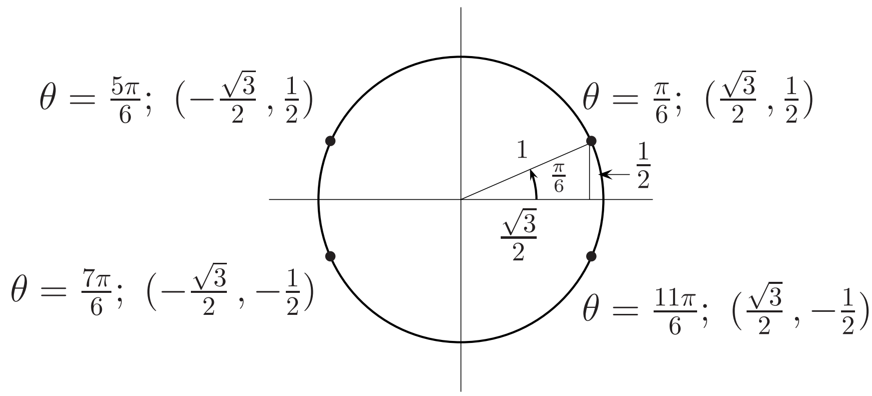



Fig. 11.6

This means $\cos\dfrac{5\pi}{6} = x$ coordinate $= -\dfrac{\sqrt{3}}{2}$, $\sin\dfrac{5\pi}{6} = y$ coordinate $= \dfrac{1}{2}$, etc. Again, this agrees with what we learned in our discussion of reference angles and quadrant sign changes in Review Topic 9d. Similar comments hold for $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{5\pi}{4}, \dfrac{7\pi}{4}$ and for $\theta = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$.

The argument is exactly the same if $\theta$ is a negative angle. For example, if $\theta = -\dfrac{\pi}{6}$, then $\theta$ has the same $x$ and $y$ coordinates as $\theta = \dfrac{11\pi}{6}$ in Fig. 11.6. This means $\cos\left(-\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$ and $\sin\left(-\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$. In a similar fasion, if $\theta = -\dfrac{5\pi}{6}$, which gives the same point $P$ as $\theta = \dfrac{7\pi}{6}$ in Fig. 11.6, then $\cos\left(-\dfrac{5\pi}{6}\right) = \dfrac{-\sqrt{3}}{2}$ and $\sin\left(\dfrac{-5\pi}{6}\right) = -\dfrac{1}{2}$.

The unit circle definition also enables us to determine when the trigonometric functions assume positive or negative values. For example, since the $y$ coordinate $> 0$ in the first and second quadrants, we conclude that $\sin\theta$ (the $y$ coordinate)$> 0$ if $0 < \theta < \pi$. If $\pi < \theta < 2\pi$, then $\sin\theta < 0$ since the $y$

coordinate is negative for these values. Similarly, $\cos\theta$ (the $x$ coordinate) is positive if $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ and negative if $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$. This gives us another way to complete Table 9d.9. in Review Topic 9d.

Periodicity is a property of the trigonometric functions that follows easily from the unit circle definition. The $x$ and $y$ coordinates of a point on the unit circle repeat themselves after one complete revolution or $2\pi$ radians. This means that $\cos\theta$ has the same values for $\theta$ and $\theta \pm 2\pi, \theta \pm 4\pi$, etc. A similar statement holds for $\sin\theta$. We say that $\cos\theta$ and $\sin\theta$ are periodic with period $2\pi$. Thus,

$$\cos\left(\frac{13\pi}{3}\right) = \cos\left(\frac{\pi}{3} + 4\pi\right) = \cos\frac{\pi}{3} = \frac{1}{2}, \text{ and}$$

$$\cos\left(\frac{17\pi}{3}\right) = \cos\left(\frac{5\pi}{3} + 4\pi\right) = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}.$$

In each case, we rewrote the argument in terms of multiples of $2\pi$ and utilized the peridicity property.

The situation regarding periodicity is different for $\tan\theta$. Careful examination shows that the values of $\tan\theta$ start to repeat themselves after $\theta$ has traveled $\pi$ radians. That is, the values of $\tan\theta$ for $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$ are the same as those for $\dfrac{\pi}{2} < \theta < \dfrac{3\pi}{2}$. In other words, $\tan\theta$ is periodic with period $\pi$.

PRACTICE PROBLEMS for Topic 11 – (Unit Circle Definition of the Trigonometric Functions)

11.1 Complete Table 11.4.

11.2 What quadrant has $\tan\theta$ negative and $\cos\theta$ positive?

11.3 Without using a calculator, find exact values for the following.

a) $\sin 2\pi$, b) $\sin 5\pi$, c) $\tan\left(\frac{11\pi}{4}\right)$,

d) $\cos\left(\frac{17\pi}{6}\right)$, e) $\sin\left(-\frac{11\pi}{6}\right)$.

ANSWERS to SAMPLE PROBLEMS (Topic 11–Unit Circle Definition of the Trigonometric Functions)

11.1

| $\theta$ | $0$ | $\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{2}$ |
|---|---|---|---|---|
| $\sin\theta$ | $0$ | $1$ | $0$ | $-1$ |
| $\cos\theta$ | $1$ | $0$ | $-1$ | $0$ |
| $\tan\theta$ | $0$ | und | $0$ | und |

"und" means undefined.

Table 11.4

11.2. Fourth quadrant.

11.3 a) 0; b) 0;

c) $\tan\left(\dfrac{11\pi}{4}\right) = \tan\left(\dfrac{3\pi}{4} + 2\pi\right) = \tan\left(\dfrac{3\pi}{4}\right) = -1;$

d) $\cos\left(\dfrac{17\pi}{6}\right) = \cos\left(\dfrac{5\pi}{6} + 2\pi\right) = \cos\left(\dfrac{5\pi}{6}\right) = \dfrac{1}{2};$

e) $\sin\left(\dfrac{-11\pi}{6}\right) = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}.$

Beginning of Topic Skills Assessment