MATH 150 – TOPIC 16

TRIGONOMETRIC EQUATIONS

In calculus, you will often have to find the zeros or x-intercepts of a function. That is, you will have to determine the values of x that satisfy the equation $f(x) = 0$. This section reviews some techniques for solving $f(x) = 0$ when $f(x)$ is a trigonometric function. A good grasp of Review Topics 12, 13, 14, and 15 will prove to be very useful. Also, remember that after you solve an equation you can always check your answer.

We first consider the following equation.

Solve for x: $\quad \sin x = \dfrac{1}{2}, \quad 0 \le x \le 2\pi.$

The graph of $\sin x$ for $0 \le x \le 2\pi$ and the line $y = \dfrac{1}{2}$ appear below.

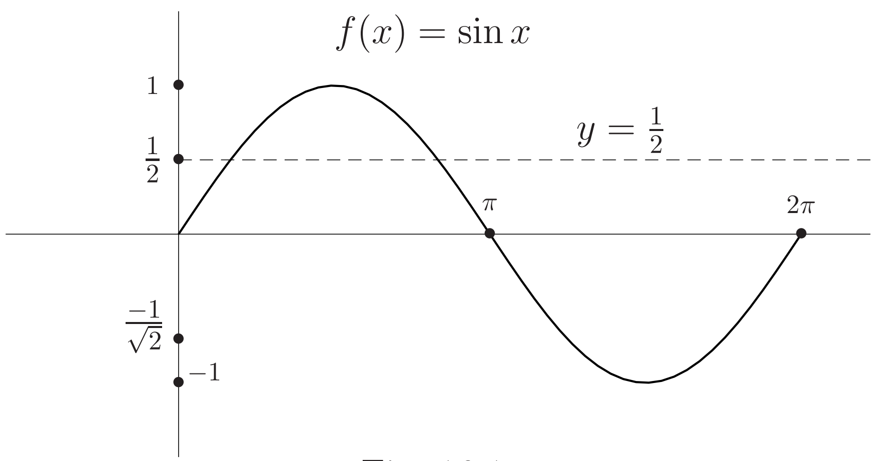

Fig. 16.1.

When $0 \le x \le 2\pi$, we see that there are two values of x where $\sin x = \dfrac{1}{2}$; namely, $x = \dfrac{\pi}{6}$ and $x = \dfrac{5\pi}{6}$. Similarly, the equation $\sin x = -\dfrac{1}{\sqrt{2}}$ has solutions $x = \dfrac{5\pi}{4}$ and $x = \dfrac{7\pi}{4}$. The equation $\sin x = 2$ has no solutions since the range of $\sin x = [-1, 1]$ (i.e., the outputs must be between -1 and 1).

A slightly more difficult equation to solve is the following.

Solve for x: $\sin x = .4, \quad 0 \le x \le 2\pi.$

In the previous equations we were able to solve for x exactly. Now we must use a calculator, which yields that $x = \arcsin(.4) = .41$ radians. The angle .41 radians is in the first quadrant. Fig. 16.1 implies there is a second value of x where $\sin x = .4$. Using the concept of the reference angle (Review Topic 9a), this second value lies in the second quadrant and $= (\pi - .41)$ or 2.73 radians.

Practice Problem

16.1. Solve the following equations where $0 \le x \le 2\pi$.

a) $\cos x = \frac{1}{2}$

b) $\sin x = 0$

c) $\cos x = .65$ (use a calculator)

Next, consider a problem of the form:

$$\sin 2x = \frac{1}{2}, \quad 0 \le x \le 2\pi.$$

Although we plug in x, the sine "sees" $2x$. In other words, if $0 \le x \le 2\pi$, the sine "sees" $0 \le 2x \le 4\pi$. On this extended interval $[0, 4\pi]$, $\sin(\ \ ) = \frac{1}{2}$ at $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi$. This means $2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{6} + 2\pi, \frac{5\pi}{6} + 2\pi$. Thus,

$$x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{\pi}{12} + \pi \text{ or } \frac{13\pi}{12}, \frac{5\pi}{12} + \pi \text{ or } \frac{17\pi}{12}.$$

Practice Problem

16.2. Solve the following equation.

$$\cos 2x = 1, \quad 0 \le x \le 2\pi.$$

Many times trigonometric equations will be quadratic in form. Consider the examples below.

$$\begin{aligned} 2\cos^2 x = 3\cos x - 1 &\qquad \text{versus} \qquad 2r^2 = 3r - 1 \\ \sin^2 x - \sin x = 0 &\qquad \text{versus} \qquad r^2 - r = 0 \end{aligned}$$

This suggests solving the trigonometric equation as if it were a quadratic equation. Let's do the first example.

Solve: $\quad 2\cos^2 x = 3\cos x - 1, \quad 0 \le x \le 2x.$

$$\begin{aligned} 2\cos^2 x - 3\cos x + 1 = 0 &\qquad \text{Get 0 on one side} \\ (2\cos x - 1)(\cos x - 1) = 0 &\qquad \text{Factor} \end{aligned}$$

$$\begin{array}{r|l} 2\cos x - 1 = 0 & \cos x - 1 = 0 \qquad \text{Set each factor} = 0 \\ 2\cos x = 1 & \cos x = 1 \\ \cos x = \dfrac{1}{2} & x = 0, 2\pi \\ x = \dfrac{\pi}{3}, \dfrac{5\pi}{3} & \end{array}$$

Solutions: $x = 0, \dfrac{\pi}{3}, \dfrac{5\pi}{3}, 2\pi.$

Practice Problem

16.3. Solve the following equations.

a) $\sin^2 x - \dfrac{3}{4} = 0, \quad 0 \le x \le 2\pi$

b) $\tan^2 x + \tan x = 0, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

Trigonometric identities can be involved in solving trigonometric equations. Consider the following problem.

Solve: $\quad \cos 2x + \cos x = 0, \quad 0 \le x \le 2\pi.$

The fact that one function has argument $2x$ and the other function has argument x suggests that we try to get the same argument in both functions. Using the identity $\cos 2x = 2\cos^2 x - 1$, we proceed as follows.

$$\cos 2x + \cos x = 0, \quad 0 \le x \le 2\pi$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$
$$\begin{array}{r|c} 2\cos x - 1 = 0 & \cos x + 1 = 0 \\ \cos x = \dfrac{1}{2} & \cos x = -1 \\ x = \dfrac{\pi}{3}, \dfrac{5\pi}{3} & x = \pi \end{array}$$

Solutions: $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$.

Question: Why did we choose the identity $\cos 2x = 2\cos^2 x - 1$ in the preceding problem? There are other ways to write $\cos 2x$. Why wouldn't they work?

As another example, consider

$$\sin x \cos x = -\frac{1}{2}, \quad 0 \le x \le 2\pi.$$

Since $\sin 2x = 2\sin x \cos x$, we have

$$\frac{\sin 2x}{2} = -\frac{1}{2}, \text{ or}$$
$$\sin 2x = -1, \quad 0 \le x \le 2\pi.$$

Notice that the sine "sees" $2x$ as in an earlier example. This means that when $0 \le x \le 2\pi$, the sine "sees" $0 \le 2x \le 4\pi$. Thus $\sin(2x) = -1$ when $2x = \dfrac{3\pi}{2}$ or $2x = \dfrac{7\pi}{2}$, which implies $x = \dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$.

Practice Problem

16.4. Solve the following equations.

a) $1 + \sin x = 2\cos^2 x, \quad 0 \le x \le 2\pi.$

b) $\tan x = 2\sin x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}.$

The next example is a bit tricky.

Solve: $\cos x + \sin x = 0, \quad 0 \le x \le 2\pi.$

In previous problems, it was helpful to get 0 on one side and then work with the other side. For this equation, we instead move $\sin x$ to the right hand side and obtain

$$\cos x = -\sin x, \quad 0 \le x \le 2\pi.$$

Now we remember that $\cos x$ and $\sin x$ are coordinates of a point P as it moves around the unit circle (Review Topic 11). So, the equation $\cos x = -\sin x$ is equivalent to asking when the coordinates of P have the same absolute value but differ in sign. A moment's thought reveals that x must be $\dfrac{3\pi}{4}$ or $\dfrac{7\pi}{4}$.

Another solution method is as follows.

Solve: $\cos x = -\sin x, \quad 0 \le x \le 2\pi.$

Divide each side by $(-\cos x)$ and obtain

$$-1 = \tan x, \quad 0 \le x \le 2\pi, \quad x \ne \frac{\pi}{2}, \frac{3\pi}{2}.$$

NOTE: We must exclude $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$ because $\tan x$ is not defined for these values. Also, $\dfrac{\pi}{2}$ or $\dfrac{3\pi}{2}$ do not solve the original equation. Thus we have

$$\tan x = -1, \quad 0 \le x \le 2\pi, \quad x \ne \frac{\pi}{2}, \frac{3\pi}{2}.$$

Solutions: $x = \frac{3\pi}{4}, \ \frac{7\pi}{4}$.

Practice Problem

16.5. Solve the equation $\sin x - \cos x = 0, \quad 0 \leq x \leq 2\pi$.

In the previous problems, the equations were solved on some specified interval. However, the interval can be unrestricted. For example,

Solve: $\quad \sin x = \frac{1}{2}$, for any x.

For problems like this, we must remember that the sine function is periodic with period 2π. So we first solve the equation on the interval $0 \leq x < 2\pi$, and then we add multiples of 2π. The solution is:

$$x = \frac{\pi}{6} + 2k\pi \text{ or } \frac{5\pi}{6} + 2k\pi, \text{ for any integer } k$$

(k can be positive or negative).

Practice Problem

16.6. Solve the following equations where x is arbitrary.

a) $\cos^2 x - \frac{\cos x}{2} = 0$.

b) $3\tan^2 x - 1 = 0$.

ANSWERS to PRACTICE PROBLEMS (Topic 16–Trigonometric Equations)

16.1. a) $x = \frac{\pi}{3}, \ \frac{5\pi}{3}$

b) $x = 0, \ \pi, \ 2\pi$

c) Solve $\cos x = .65$, $0 \le x \le 2\pi$.

Using arccos( ) on your calculator (in radian mode), you get $x = .863$ radians. However, $\cos x$ is also positive in the fourth quadrant where $\frac{3\pi}{2} \le x \le 2\pi$. The reference angle concept (Review Topic 9a) implies that $x = 2\pi - .863 = 5.42$ radians.

Solutions: $x = .863$ or 5.42

16.2. Solve $\cos(2x) = 1$, $0 \le x \le 2\pi$.

The cosine "sees" $(2x)$, which means $0 \le (2x) \le 4\pi$. On this interval, cos( ) = 1 when ( ) $= 0, 2\pi, 4\pi$. Thus, $(2x) = 0, 2\pi$, or 4π, which implies $x = 0, \pi$, or 2π.

16.3. a) Method 1. Factor the left hand side and obtain

$$\left(\sin x - \frac{\sqrt{3}}{2}\right)\left(\sin x + \frac{\sqrt{3}}{2}\right) = 0, \quad 0 \le x \le 2\pi.$$

$$\begin{array}{r|r} \sin x - \frac{\sqrt{3}}{2} = 0 & \sin x + \frac{\sqrt{3}}{2} = 0 \\ \sin x = \frac{\sqrt{3}}{2} & \sin x = -\frac{\sqrt{3}}{2} \\ x = \frac{\pi}{3}, \frac{2\pi}{3} & x = \frac{4\pi}{3}, \ \frac{5\pi}{3} \end{array}$$

Solutions: $x = \frac{\pi}{3}, \ \frac{2\pi}{3}, \ \frac{4\pi}{3}, \ \frac{5\pi}{3}$

Method 2. Solve for $\sin^2 x$.

$$\sin^2 x = \frac{3}{4}$$

$$\sin x = \pm\frac{\sqrt{3}}{2}. \quad \text{So,}$$

$$\sin x = \frac{\sqrt{3}}{2} \;\Big|\; \sin x = -\frac{\sqrt{3}}{2}$$

Now, continue as above.

b) Solve $\tan^2 x + \tan x = 0$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$\tan x(\tan x + 1) = 0$$

$$\begin{array}{r|c} \tan x = 0 & \tan x + 1 = 0 \\ x = 0 & \tan x = -1 \\ & x = -\frac{\pi}{4} \end{array}$$

Solutions: $x = -\frac{\pi}{4}, 0$.

16.4. a) Solve

$$\begin{aligned} 1 + \sin x &= 2\cos^2 x,\ 0 \le x \le 2\pi \\ &= 2(1 - \sin^2 x) \\ &= 2 - 2\sin^2 x \end{aligned}$$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$$\begin{array}{r|c} 2\sin x - 1 = 0 & \sin x + 1 = 0 \\ \sin x = \frac{1}{2} & \sin x = -1 \\ x = \frac{\pi}{6}, \frac{5\pi}{6} & x = \frac{3\pi}{2} \end{array}$$

Solutions: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$.

b) Solve $\tan x = 2\sin x$, $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

$$\frac{\sin x}{\cos x} - 2\sin x = 0$$
$$\sin x\left(\frac{1}{\cos x} - 2\right) = 0$$

$$\begin{array}{c|c}
\sin x = 0 & \dfrac{1}{\cos x} - 2 = 0 \\
x = 0 & \dfrac{1}{\cos x} = 2 \\
 & \cos x = \dfrac{1}{2} \\
 & x = -\dfrac{\pi}{3}, \dfrac{\pi}{3}
\end{array}$$

Solutions: $x = -\dfrac{\pi}{3}, 0, \dfrac{\pi}{3}$.

16.5. Solve $\sin x - \cos x = 0$, $0 \le x \le 2\pi$.

Method 1: $\sin x = \cos x$; $0 \le x \le 2\pi$.

Since $\cos x$ and $\sin x$ are the x and y coordinates of a point P on the unit circle, this question is equivalent to asking when the coordinates have the same value. The answer is when P is in the first and third quadrants where $x = \dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$, respectively.

Method 2: Divide by $\cos x$ and obtain

$$\tan x - 1 = 0, \quad 0 \le x \le 2\pi, \quad x \ne \frac{\pi}{2}, \ \frac{3\pi}{2}$$
$$\tan x = 1$$
$$x = \frac{\pi}{4}, \ \frac{5\pi}{4}.$$

16.6. a) Solve $\cos^2 x - \dfrac{\cos x}{2} = 0$, where x is arbitrary.

$$\cos x \left(\cos x - \frac{1}{2}\right) = 0$$

$\cos x = 0$	$\cos x - \frac{1}{2} = 0$
$x = \frac{\pi}{2} + 2k\pi$, or	$\cos x = \frac{1}{2}$
$x = \frac{3\pi}{2} + 2k\pi$	$x = \frac{\pi}{3} + 2k\pi,\ \frac{5\pi}{3} + 2k\pi$
Equivalently,	
$x = \frac{\pi}{2} + k\pi$	

Solutions: $x = \dfrac{\pi}{2} + k\pi,\ \dfrac{\pi}{3} + 2k\pi,\ \dfrac{5\pi}{3} + 2k\pi$, for any integer k.

b) Solve $3\tan^2 x - 1 = 0$.

$$\tan^2 x = \frac{1}{3}$$

$$\tan x = \pm\frac{1}{\sqrt{3}}$$

$\tan x = \frac{1}{\sqrt{3}}$	$\tan x = -\frac{1}{\sqrt{3}}$
$x = \frac{\pi}{6} + k\pi$	$x = -\frac{\pi}{6} + k\pi$

Solution: $x = \pm\dfrac{\pi}{6} + k\pi$, for any integer k. (Remember that $\tan x$ has period π. This means that we only need to add multiples of π, not 2π.)

Beginning of Topic Skills Assessment