MATH 150 – TOPIC 3

FUNCTIONS: COMPOSITION AND DIFFERENCE QUOTIENTS

I. Composition of Functions

HAPPY BIRTHDAY... OK so in all likelihood today isn't your birthday. Let's assume it is and you've been given a present wrapped in a large box. You open the box and surprise!!!, inside is another box. Undaunted you open it and discover yet another box. This may appear to be a lesson in accepting disappointment. Unbeknownst to you, though, you are experiencing composite function behavior.

HUH?!?! Let us explain.

The composite function is like having one function contained inside another. When you see $\sqrt{2x-1}$, you probably think of it as just another function, but it's something more. It's a composite (one function inside another). Here's why.

Suppose $f(x) = \sqrt{x}$ and $g(x) = 2x-1$ (think of $f(\quad) = \sqrt{(\quad)}$ as the larger box). Now place $2x-1$ inside. Mathematically this is written as $f(g(x))$ and we call it "f composite g". To form a composite, try the following:

$$f(g(x)) = f(\quad) = \sqrt{(\quad)}.$$

Now fill in the blanks with the $g(x)$ representation and you get
$f(g(x)) = f(2x-1) = \sqrt{2x-1}$.

NOTE: just like $f(3)$ means to input 3 into f, $f(g(x))$ means to input g into f.

Unlike boxes, any function can be placed inside another. From above,
$g(x) = 2x-1$. This really means $g(\quad) = 2(\quad) - 1$. If $f(x) = \sqrt{x}$, we have

$$g(f(x)) = g(\sqrt{x}) = 2(\sqrt{x}) - 1 = 2\sqrt{x} - 1.$$

Alternative notation for composite functions:

$$g(f(x)) = g \circ f, \quad f(g(x)) = f \circ g.$$

Let's try a different practice drill. Suppose $h(x) = \sin(2x)$. Can you state functions for f and g so that $h = f(g(x))$?

Solution: inner function is $2x$, which is g,
outer function is $\sin x = \sin(\quad)$, which is f.

Check: $f(g(x)) = f(\quad) = \sin(\quad)$.
Now insert $2x$ and get $f(g(x)) = f(2x) = \sin 2x$.

NOTE: This is the most obvious choice for f and g but not the only one. Let us try $f(x) = \sin(x+1)$ and $g(x) = 2x - 1$.

Check: $f(g(x)) = f(\quad) = \sin[(\quad) + 1]$,
so $f(g(x)) = f(2x-1) = \sin[(2x-1)+1] = \sin 2x$.

What's so special about composite functions? One day in Calc I your instructor will throw a birthday party for the entire class, and his present to you will be called the CHAIN RULE.

Exercise: Practice Problem 3.1.

II. Difference Quotients

A. Introduction

In algebra, rate of change is introduced in its most basic form by finding the slope of a line using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. This "formula" can also be called a Difference Quotient. In calculus, rates of change (both average and instantaneous) are found using function forms of difference quotients.

Let's start with a few examples of the algebra in difference quotients.

Example: Let $g(x) = x^2 + 1$. Evaluate and simplify the difference quotient $\dfrac{g(x) - g(2)}{x - 2}$, $x \neq 2$.

Solution: Using $g(x) = x^2 + 1$ and $g(2) = 5$,

$$\frac{g(x) - g(2)}{x - 2} = \frac{(x^2 + 1) - 5}{x - 2} = \frac{x^2 - 4}{x - 2} = x + 2.$$

Example: Let $g(x) = \dfrac{1}{x - 2}$. Evaluate and simplify the difference quotient $\dfrac{g(x + \Delta x) - g(x)}{\Delta x}$, $\Delta x \neq 0$ [Δx represents the "change" in x.]

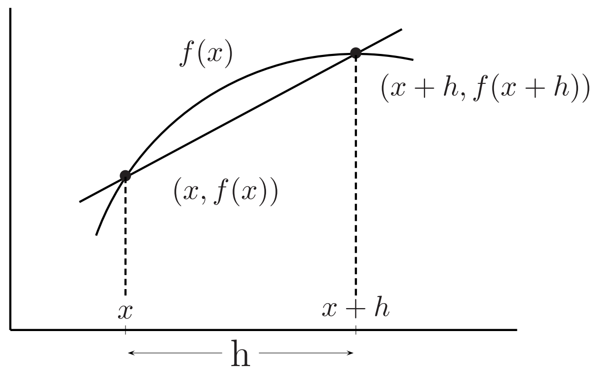
Remember: $g(\quad) = \dfrac{1}{(\quad) - 2}$.

$$\begin{aligned}\frac{g(x + \Delta x) - g(x)}{\Delta x} &= \frac{\dfrac{1}{(x + \Delta x) - 2} - \dfrac{1}{x - 2}}{\Delta x} \\ &= \frac{(x - 2) - (x + \Delta x - 2)}{(x + \Delta x - 2)(x - 2)} \cdot \frac{1}{\Delta x} \\ &= \frac{-\Delta x}{(x + \Delta x - 2)(x - 2)(\Delta x)} \\ &= \frac{-1}{(x + \Delta x - 2)(x - 2)}\end{aligned}$$

B. Application

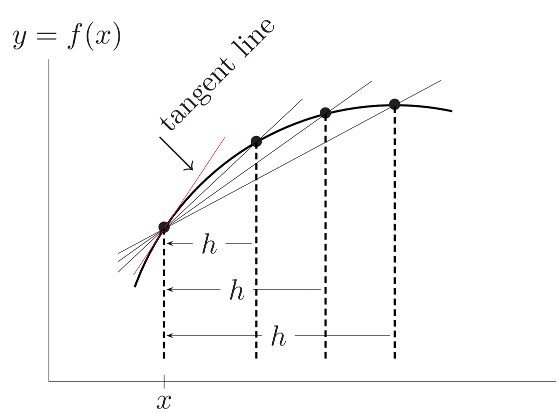
How does a difference quotient measure rate of change (or slope)?

Suppose we start with a function $f(x)$ and draw a line (called a secant line) connecting two of its points.



$$m = \frac{y_2 - y_1}{x - 2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

Notice that the slope is represented by a difference quotient and is referred to as the average rate of change. Now suppose the distance between these two points is shortened by decreasing the size of h.



Conclusion: The two points get so close together they almost concide. The resulting line is now a tangent line whose slope measures the instantaneous rate of change. You will come to know this slope by the term "derivative".

PRACTICE PROBLEMS for Topic 3 – Functions and Difference Quotients

3.1 a) Suppose $f(x) = x^2 - 2x$ and $g(x) = \dfrac{1}{x}$. Form the following compositions:

i) $f(g(x))$ ii) $g(f(x))$ iii) $f(f(x))$

b) Suppose $f(x) = \sin x$ and $g(x) = \sqrt{x}$. Form the following compositions:

i) $f(g(x))$ ii) $g(f(x))$

c) Suppose $y = \sqrt{\sin 3x}$. Select functions for f, g, and h so that $y = f(g(h))(x)$. Check your results.

Return to Review Topic Answers

3.2 Find the rate of change (slope) of the line containing the given points.

a) $(2, -5)$ and $(4,1)$

b) $(3, -4)$ and $(3, -1)$

c) $(x, f(x))$ and $(x + h, f(x + h))$ Answers

3.3 Evaluate and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$, when:

a) $f(x) = x^2 - 3x$;

b) $f(x) = \sqrt{x-2}$ (Rationalize the numerator). Answers

3.4 Evaluate and simplify the difference quotient $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$, $\Delta x \neq 0$, when $f(x) = \dfrac{1}{2+x}$. Answer

ANSWERS to PRACTICE PROBLEMS
(Topic 3 – Functions and Difference Quotients)

3.1 a) i) $\left(\dfrac{1}{x}\right)^2 - 2\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2} - \dfrac{2}{x}$

ii) $\dfrac{1}{x^2 - 2x}$

iii) $(x^2 - 2x)^2 - 2(x^2 - 2x) = (x^2 - 2x)(x^2 - 2x - 2)$

b) i) $\sin\sqrt{x}$

ii) $\sqrt{\sin x}$

c) inner: $h(x) = 3x$
middle: $g(x) = \sin x$
outer: $f(x) = \sqrt{x}$

CHECK: $f(g(h))(x) = f(g(\underset{\substack{\uparrow \\ h}}{3x})) = f(\underset{\substack{\uparrow \\ g}}{\sin} 3x) = \underset{\substack{\uparrow \\ f}}{\sqrt{\sin 3x}}$

Return to Problem

3.2 a) $m = 3$ b) m is undefined c) $m = \dfrac{f(x+h) - f(x)}{h}$

Return to Problem

3.3 a) $$\frac{(x+h)^2-3(x+h)-(x^2-3x)}{h}$$
$$=\frac{2xh+h^2-3x}{h}$$
$$=2x+h-3$$

b) $$\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\cdot\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}$$
$$=\frac{(\sqrt{x+h-2})^2-(\sqrt{x-2})^2}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$
$$=\frac{x+h-2-(x-2)}{h(\sqrt{x+h-2}+\sqrt{x-2})}$$
$$=\frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$$

Return to Problem

3.4 $$\frac{\dfrac{1}{2+(x+\Delta x)}-\dfrac{1}{2+x}}{\Delta x}$$
$$=\frac{2+x-(2+x+\Delta x)}{(2+x+\Delta x)(2+x)\Delta x}$$
$$=\frac{1}{(2+x+\Delta x)(2+x)}$$

Return to Problem