MATH 150 – TOPIC 6

GRAPHING TECHNIQUES FOR POLYNOMIAL AND RATIONAL FUNCTIONS

I. Polynomial Functions.

Polynomial functions are the easiest functions to graph. They are well-behaved, smooth, no pieces or asymptotes, and have domains that include ‘all reals’. Below a process is presented for graphing a polynomial function.

Example: Graph $f(x) = x^3 + 4x^2 - x - 4$; include all intercepts.

Solution: First we will find intercepts.

y-intercept: $f(0) = -4$
x-intercept: Solve $f(x) = 0$.

$$\begin{aligned} f(x) &= x^2(x+4) - (x+4) \\ &= (x^2-1)(x+4) \\ &= (x+1)(x-1)(x+4) \end{aligned}$$

x-intercepts: $x = \pm 1$ and -4

These are also called the “zeroes” of f.

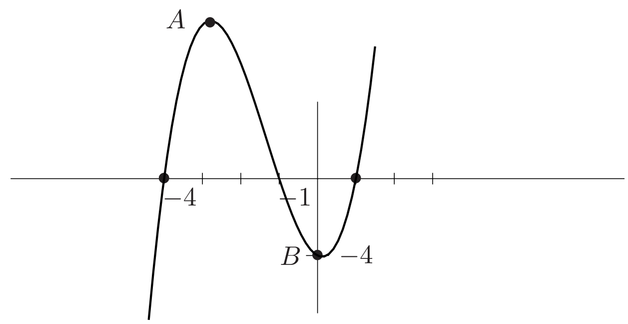
Now we will need to find where $f > 0$ and $f < 0$. If you recall (Review Topic 5, Function Behavior), this can be accomplished with a table of signs.

		-4		-1		1	
f	$-$		$+$		$-$		$+$

As a result,

$$f < 0 \text{ on } (-\infty, -4) \cup (-1, 1)$$
$$f > 0 \text{ on } (-4, -1) \cup (1, \infty).$$

Finally, we can proceed to graph f.



The turning points at A and B are also of interest and can easily be found using methods taught in calculus. In general, a polynomial of degree n has at most $n-1$ turning points and at most n x-intercepts.

II. Rational Functions and Asymptotes

Think of these as 'fraction functions'. Here are 2 examples:

$$f(x) = \frac{2}{x-2}, \qquad g(x) = \frac{x}{x^2 - 2x - 3}.$$

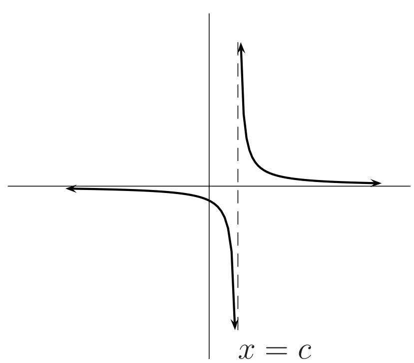
Part of the difficulty in graphing such functions is due to lines called asymptotes. Asymptotes can be vertical, horizontal, or occasionally oblique (slanted).

A. Vertical Asymptotes (VA)

DEF: The line $x = c$ is a vertical asymptote for the graph of a function if (a) $f(c)$ is undefined and (b) $f(x) \to \infty$ or $-\infty$ as x approaches c from the right (c^+) or from the left (c^-).

What does that mean?!?

Vertical asymptotes restrain a function by acting like fences. As the function nears an asymptote it changes direction to avoid a crash. Eventually it's path becomes almost vertical as shown below.



"What causes this fence-like behavior?!?"

The answer may come from examining the behavior of a function "near" a value where f is undefined.

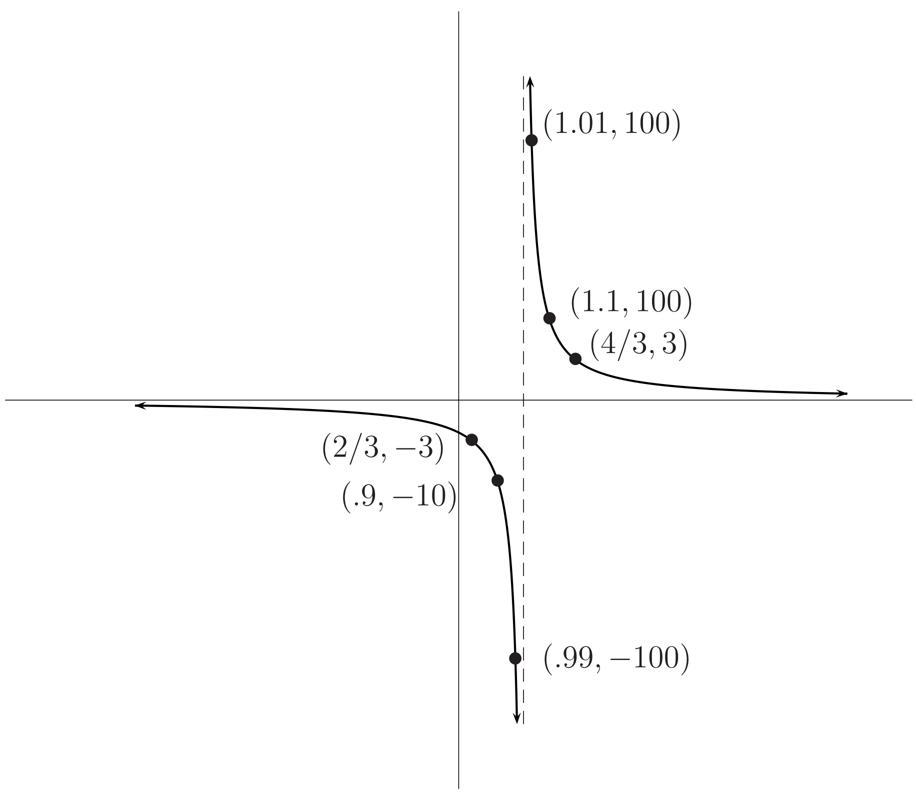
Example 1: Discuss the behavior of $f(x) = \dfrac{1}{x-1}$ as $x \to 1$.

$x \to 1^-$ (from the left) $\qquad$ $x \to 1^+$ (from the right)

x	.9	.99	.999	1	1.001	1.01	1.1
$f(x) = \dfrac{1}{x-1}$	$\dfrac{1}{-.1} = -10$	$\dfrac{1}{-.01} = -100$	$\dfrac{1}{-.001} = -1000$	undefined	$\dfrac{1}{.001} = 1000$	$\dfrac{1}{.01} = 100$	$\dfrac{1}{.1} = 10$

$f(x) \to -\infty$ $\qquad$ $x = 1$ is V.A. $\qquad$ $f(x) \to \infty$

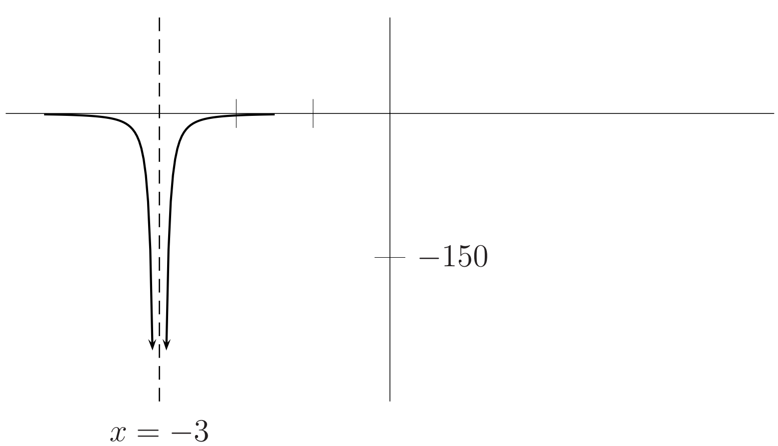
Comments: Since the numerator remains constant, the denominator must be the key. As we input values close to 1, denominators of f get smaller. This causes outputs, depending on sign, to head toward ∞ or $-\infty$. Such behavior is often referred to as "unbounded." This makes $x = 1$ a vertical asymptote. Here is the graph of f.



Example 2: Using $f(x) = \dfrac{2x}{(x+3)^2}$, make a table of values and discuss the behavior of f as $x \to -3$.

	x	$f(x) = \dfrac{2x}{(x+3)^2}$	
	-3.2	$\dfrac{-6.4}{(-.2)^2} = -160$	
$x \to -3^-$	-3.1	$\dfrac{-6.2}{(-.1)^2} = -620$	$f(x) \to -\infty$
	-3.01	$\dfrac{-6.02}{(-.01)^2} = -60200$	
	-3	undefined	
	-2.99	$\dfrac{-5.98}{(.01)^2} = -59800$	
$x \to -3^+$	-2.9	$\dfrac{-5.8}{(.1)^2} = -580$	$f(x) \to -\infty$
	-2.8	$\dfrac{-5.6}{(.2)^2} = -140$	

Comments: This time both numerator and denominator are changing. The change in the denominator is of greater importance. Dividing a nonzero number by a small positive number causes f to be unbounded as $x \to -3$. Thus $x = -3$ is a vertical asymptote. Here is a graph of f near $x = -3$.



Exercise.

a) Complete the table below for $f(x) = \dfrac{x^2 - 9}{x + 3}$.

x	$f(x)$
-3.2	
-3.1	
-3.01	
-3	
-2.99	
-2.9	
-2.8	

Answer

Based on the behavior of f as $x \to -3$, indicate the following:

b) For what values of x is f undefined? Answer

c) Is the function f unbounded as $x \to -3$? Answer

d) Is $x = -3$ a vertical asymptote? Explain. Answer

e) Other than the table, is there any way to predict $x = -3$ is not a vertical asymptote? Answer

f) Sketch the graph near $x = -3$. Answer

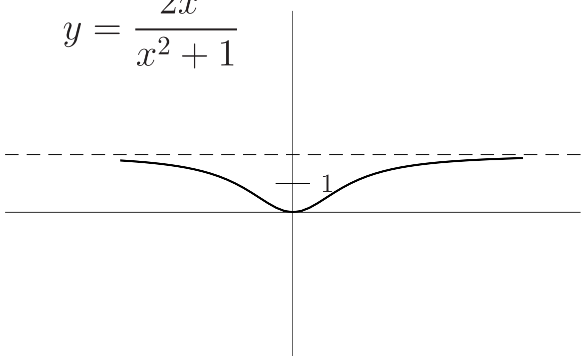
B. Horizontal Asymptotes (HA)

Horizontal asymptotes are not as restrictive as vertical asymptotes. Occasionally they are crossed.

DEF: The line $y = k$ is the horizontal asymptote of a function if $f(x) \to k$ as $x \to \infty$ or $x \to -\infty$.

Again, an example might help.

$$y = \frac{2x^2}{x^2+1}$$



The line $y = 2$ is a HA.
That means $f(x) \to 2$ as
$x \to -\infty$ and as $x \to \infty$

To find horizontal asymptotes for a rational function let

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1}x^{n-1} \cdots a_0}{b_n x^n + b_{n-1}x^{n-1} + \cdots b_0};$$

i) If $p(x)$ is of greater degree than $q(x)$, there are no horizontal asymptotes.

ii) If $p(x)$ and $q(x)$ are equal in degree, then $y = \dfrac{a_n}{b_n}$ is the HA.

iii) If $p(x)$ is of lesser degree than $q(x)$, $y = 0$ is the HA.

Example: Find the VA and HA of $f(x) = \dfrac{2x^2}{x(x-1)}$.

VA: $x = 0$, $x = 1$ ($f(0)$ and $f(1)$ are undefined)
HA: Since the numerator and denominator are both of degree 2, then $y = \dfrac{a}{b} = 2$ is the HA.

Now we can proceed to graphing a rational function.

Example: Graph $f(x) = \dfrac{x-2}{x^2-2x-3}$; include all intercepts and asymptotes. We first must find intercepts and asymptotes.

Intercepts:

$f(0) = \dfrac{2}{3} \Rightarrow (0, 2/3)$

$f(x) = 0$ when $x = 2 \Rightarrow (2, 0)$

Remember: A rational function will equal 0 only when its numerator equals 0.

Asymptotes:

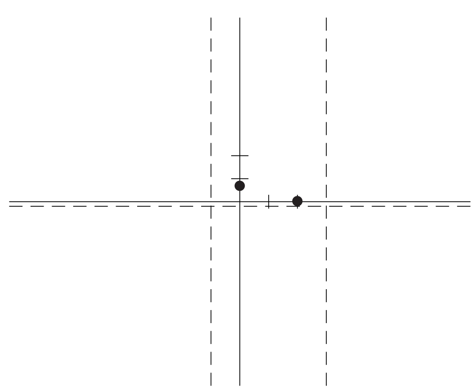
VA: $x = 3, -1$

HA: $y = 0$ (numerator is of lesser degree)

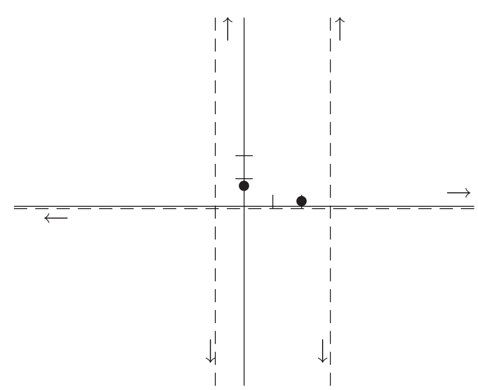
We will probably still want to know when f is above and below the x-axis. Make a sign table.

		-1		2		3	
f	$-$		$+$		$-$		$+$

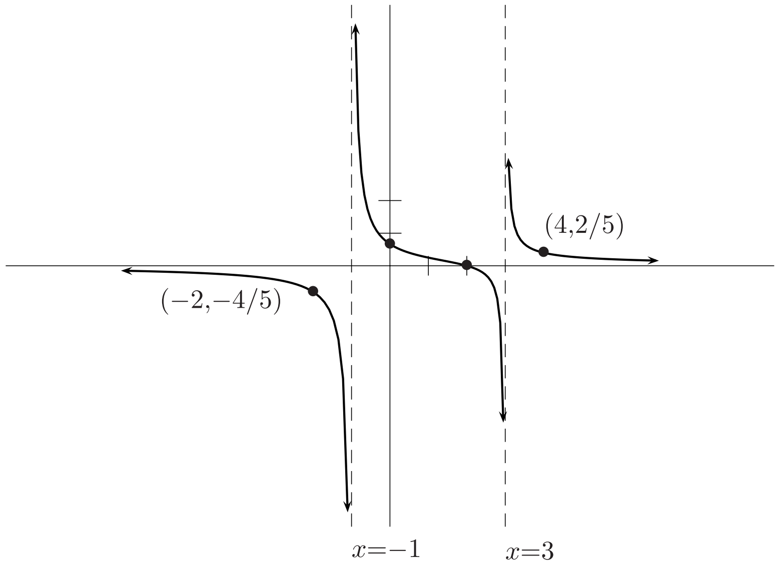
Here are 3 stages of the graph.



asymptotes and intercepts



$f > 0$ and $f < 0$ while approaching asymptotes



Here again is the asymptotic behavior in more formal notation.

$$\begin{array}{llll} f(x) \to -\infty & \text{as} & x \to -1^- & \text{(from the left)} \\ f(x) \to \ \ \infty & \text{as} & x \to -1^+ & \text{(from the right)} \\ f(x) \to -\infty & \text{as} & x \to 3^- & \\ f(x) \to \ \ \infty & \text{as} & x \to 3^+ & \\ f(x) \to 0 & \text{as} & x \to \pm\infty & \end{array}$$

PRACTICE PROBLEMS for Topic 6 – Graphing Techniques for Polynomial and Rational Functions

6.1. Find the intercepts and asymptotes for each of the following:

a) $y = x^3 + x$

b) $y = \dfrac{2}{x^3 + x}$

c) $y = \dfrac{-2x^2}{x^2 - x - 6}$

d) $f(x) = (x^2 + 2x)(x - 2)^2$

e) $f(x) = \dfrac{x + 2}{x^2 + x - 6}$

Answers

6.2. Find intervals where $f(x) < 0$ and $f(x) > 0$.

a) $f(x) = (x^2 + 2x)(x - 2)^2$

b) $f(x) = \dfrac{x + 2}{x^2 + x - 6}$ Answers

6.3. Graph each of the following. Indicate intercepts and asymptotes. Also indicate the behavior of f as $x \to \infty$, $x \to -\infty$ and $x \to c$ (if c is a VA).

a) $f(x) = (x^2 + 2x)(x - 2)^2$

b) $f(x) = \dfrac{x + 2}{x^2 + x - 6}$ Answers

ANSWERS to PRACTICE PROBLEMS (Topic 6 – Graphing Techniques for Polynomial and Rational Functions)

6.1. a) Intercepts $(0,0)$, $(-1,0)$; No Asymptotes.

b) No intercepts; VA: $x=0$, $x=-1$; HA: $y=0$.

c) Intercepts: $(0,0)$; VA: $x=3$, $x=-2$; HA: $y=-2$.

d) Intercepts: $(0,0)$, $(-2,0)$, $(2,0)$; No Asymptotes.

e) Intercepts: $(0,-1/3)$, $(-2,0)$; VA: $x=2$, $x=-3$; HA: $y=0$.

Return to Problem

6.2. a) $f>0$ on $(-\infty,-2)\cup(0,2)\cup(2,\infty)$
$f<0$ on $(-2,0)$

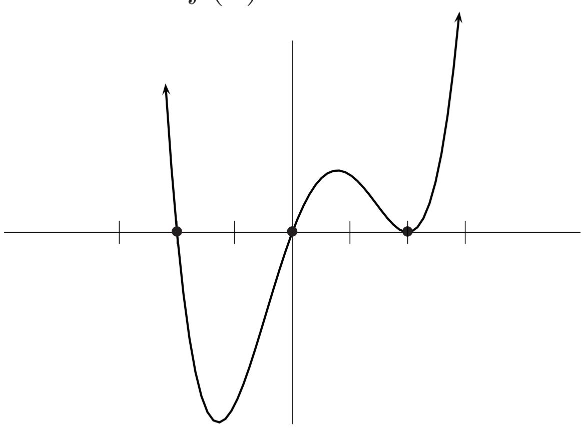
	-2		0		2	
$+$		$-$		$+$		$+$

b) $f>0$ on $(-3,-2)\cup(2,\infty)$
$f<0$ on $(-\infty,-3)\cup(-2,2)$

	-3		-2		2	
$-$		$+$		$-$		$+$

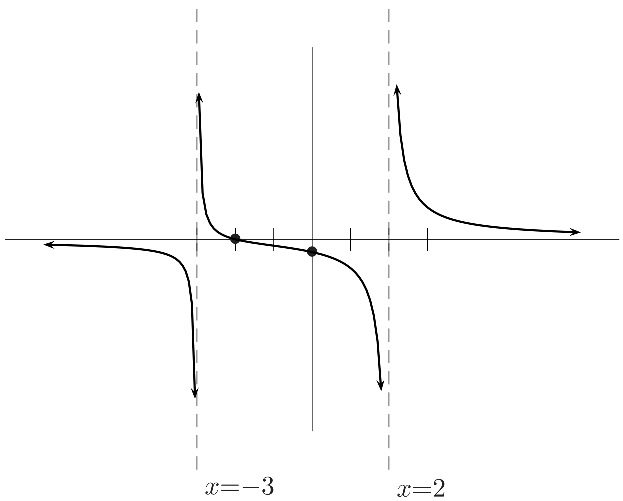
Return to Problem

6.3. a) $f(x) \to \infty$ as $x \to \infty$
$f(x) \to \infty$ as $x \to -\infty$



Intercepts: $f(-2) = 0$
$f(2) = 0$
$f(0) = 0$
Polynomials don't have asymptotes

b)



Intercepts: $f(-2) = 0$
$f(0) = -\dfrac{1}{3}$
VA: $x = -3,\ x = 2$
HA: $y = 0$

$$\begin{array}{llll}
f(x) \to -\infty & \text{as} & x \to -3^- \\
f(x) \to \infty & \text{as} & x \to -3^+ \\
f(x) \to -\infty & \text{as} & x \to 2^- \\
f(x) \to \infty & \text{as} & x \to -2^+ \\
f(x) \to 0 & \text{as} & x \to \pm\infty
\end{array}$$

Return to Problem

Beginning of Topic 150 Skills Assessment

a) Complete the table for $f(x) = \dfrac{x^2 - 9}{x + 3}$.

Based on the behavior of f as $x \to -3$, indicate the following:

b) For what values of x is f undefined?

c) Is the function f unbounded as $x \to -3$?

Answers:

a)

x	$f(x)$	
-3.2	$\dfrac{1.24}{-.2} = -6.2$	
-3.1	$\dfrac{.61}{-.1} = -6.1$	$f(x) \to -6$
-3.01	$\dfrac{.0601}{.01} = -6.01$	
-3	undefined	
-2.99	$\dfrac{-.0599}{.01} = -5.99$	
-2.9	$\dfrac{-.59}{.1} = -5.9$	$f(x) \to -6$
-2.8	$\dfrac{-1.16}{.2} = -5.8$	

b) Undefined at $x = -3$.

c) No. As $x \to -3$, $f(x) \to -6$.

Return to Review Topic

d) Is $x = -3$ a vertical asymptote? Explain.

e) Other than the table, is there any way to predict $x = -3$ is not a vertical asymptote?

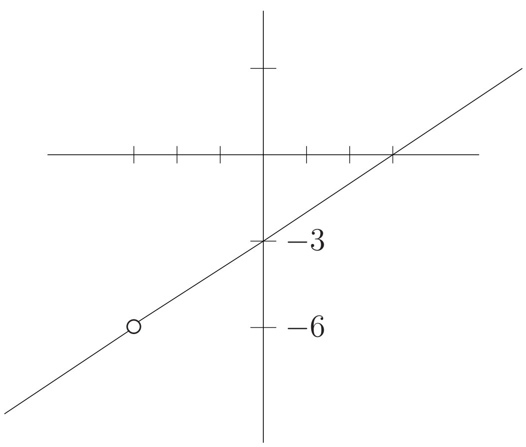
f) Sketch the graph near $x = -3$.

Answers:

d) No; although $f(-3)$ is undefined, $f(x)$ is not unbounded as $x \to -3$.

e) Look at the table. Both numerator and denominator $\to 0$ as $x \to -3$. **This always indicates that more work is needed.** After cancellation, the function $f(x) = \dfrac{x^2-9}{x+3}$ becomes the linear function $f(x) = x - 3,\ x \neq -3$. Instead of a V.A., there is a hole at $x = -3$.

f)



Final Comments: The numerical approach that we've used shows clearly what happens to function outputs as inputs approach a particular value. In calculus, you will use an analytical approach to predict such behavior and the process will be called "Finding the Limit".

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