MATH 250 – TOPIC 1

FUNCTIONS AND DIFFERENCE QUOTIENTS

We don't expect this material to cause you many problems, but just in case, let's quickly review.

I. Evaluating a function.

Exercise 1: Evaluate the function $f(x) = -x^2 + 5x$ for

a) $f(-3)$ and b) $f(x+h)$.

Answer

Exercise 2: For the function f defined by $f(x) = \dfrac{1}{x-3}$, find

a) $f\left(\dfrac{1}{a}\right)$ b) $\dfrac{1}{f(a)}$ c) $f(2+h)$ d) $f(2) + f(h)$

Remember: Write $f(x)$ as $f(\quad) = \dfrac{1}{(\quad) - 3}$. Answer

Exercise 3: Suppose $f(x) = \dfrac{x}{2^x}$, find and simplify:

a) $f(n)$ b) $f(n+1)$ c) $f(2n)$ d) $\dfrac{f(n+1)}{f(n)}$ Answer

II. Difference Quotients

A. Introduction

In algebra, rate of change is introduced in its most basic form: finding the slope of a line using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. This "formula" can also be called a Difference Quotient. In calculus, rates of change (both average and instantaneous) are found using function forms of difference quotients.

Example: Let $g(x) = x^2 + 1$. Evaluate and simplify the difference quotient $\dfrac{g(x) - g(2)}{x - 2}$, $x \neq 2$.

Solution: Using $g(x) = x^2 + 1$ and $g(2) = 5$,

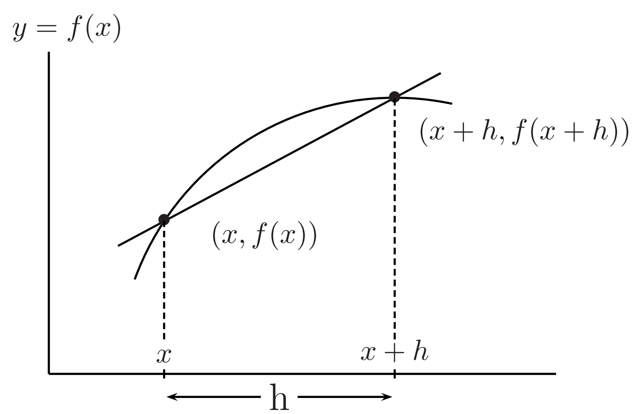
$$\frac{g(x) - g(2)}{x - 2} = \frac{(x^2 + 1) - 5}{x - 2} = \frac{x^2 - 4}{x - 2} = x + 2.$$

B. Application

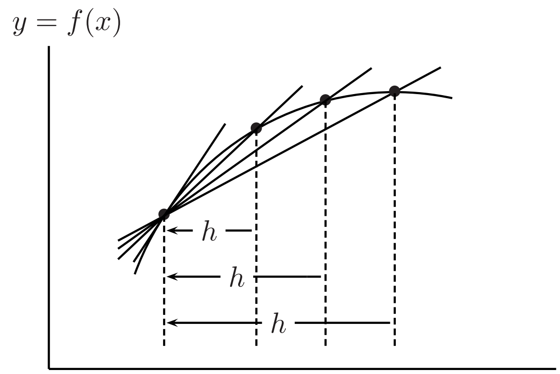
How does a difference quotient measure rate of change? Let's go back to those dark days in Calc I, the days before derivative rules ...

The lecture began with a discussion of rates of change (also known as slope). A function was drawn and the slope of the secant line was found.

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$



Notice that the slope is represented by a difference quotient. Your instructor then went further by asking, "What happens as h gets smaller?"



Conclusion: Each secant line represents a better approximation to the slope of the tangent line. Finally, by taking $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$, we are able to measure the slope of the tangent line, known as $f'(x)$.

Exercise 4: Let $f(x) = \dfrac{1}{x-2}$.

a) Evaluate and simplify the difference quotient $\dfrac{f(x+h) - f(x)}{h}$, $h \neq 0$.

b) Take $\lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ and compare the result to finding $f'(x)$ using derivative rules.

Answers

III. Exponential and Logarithmic Functions

The material on natural logs and exponentials in Calc II is similar to Calc I and includes knowledge of graphs, log properties, and limit-based behavior. See R.T. 5 in our Math 150 site for review and practice problems on any of these concepts.

Practice Problems.

1.1. Find and simplify $\dfrac{f(x+h)-f(x)}{h}$ for the following functions.

a) $f(x) = 2x - x^2$

b) $f(x) = \dfrac{1}{x^2}$

c) $f(x) = \sqrt{x-2}$; Hint: rationalize the numerator.

Answer

1.2. Let $f(x) = \dfrac{x}{(x-1)2^x}$. Find and simplify:

a) $f(n)$

b) $f(2n)$

c) $\dfrac{f(n+1)}{f(n)}$

Answer

Answers to Practice Problems

1.1. a) $$\frac{2(x+h)-(x+h)^2-(2x-x^2)}{h} = \frac{2h-2xh-h^2}{h} = 2-2x-h$$

b) $$\frac{\dfrac{1}{(x+h)^2}-\dfrac{1}{x^2}}{h} = \frac{-2xh-h^2}{x^2(x+h)^2}\cdot\frac{1}{h^2} = \frac{-2x-h}{x^2(x+h)^2}$$

c) $$\frac{\sqrt{x+h-2}-\sqrt{x-2}}{h}\cdot\frac{\sqrt{x+h-2}+\sqrt{x-2}}{\sqrt{x+h-2}+\sqrt{x-2}}$$
$$= \frac{x+h-2-(x-2)}{h(\sqrt{x+h-2}+\sqrt{x-2})} = \frac{1}{\sqrt{x+h-2}+\sqrt{x-2}}$$

If you really want to check these results, take $\lim_{h\to 0}$ and compare to $f'(x)$.

Return to Problem

1.2. a) $$\frac{n}{(n-1)2^n}$$

b) $$\frac{2n}{(2n-1)2^{2n}}$$

c) $$\frac{\dfrac{n+1}{n\cdot 2^{n+1}}}{\dfrac{n}{(n-1)2^n}} = \frac{n+1}{n\cdot 2^{n+1}}\cdot\frac{(n-1)2^n}{n}$$
$$= \frac{2n}{2^{n+1}}\left[\frac{(n+1)(n-1)}{n^2}\right] = \frac{1}{2}\left[\frac{(n+1)(n-1)}{n^2}\right]$$

Return to Problem

Evaluate the function $f(x) = -x^2 + 5x$ for

a) $f(-3)$ and b) $f(x+h)$.

Answers:

Treat each x in the function as though it were an empty set of parentheses to be filled. For our example, write $f(x)$ as

$$f(\quad) = -(\quad)^2 + 5(\quad)$$

So,

a) $f(-3) = -(-3)^2 + 5(-3)$,

and

b) $f(x+h) = -(x+h)^2 + 5(x+h)$.

Common error: $f(x+h) \neq -x^2 + 5x + h$. This is simply $f(x) + h$.

Return to Review Topic

For the function f defined by $f(x) = \dfrac{1}{x-3}$, find

a) $f\left(\dfrac{1}{a}\right)$ b) $\dfrac{1}{f(a)}$ c) $f(2+h)$ d) $f(2)+f(h)$

Answers:

a) $f\left(\dfrac{1}{a}\right) = \dfrac{1}{\left(\frac{1}{a}\right)-3} = \dfrac{a}{1-3a}$

b) $\dfrac{1}{f(a)} = \dfrac{1}{\frac{1}{a-3}} = a-3$

c) $f(2+h) = \dfrac{1}{(2+h)-3} = \dfrac{1}{h-1}$

d) $f(2)+f(h) = \dfrac{1}{2-3} + \dfrac{1}{h-3} = -1 + \dfrac{1}{h-3} = \dfrac{-h+4}{h-3} = -\dfrac{h-4}{h-3}$

Note: For most functions, $f(a+b) \neq f(a) + f(b)$.

Ex. $\sin(\alpha+\beta) \neq \sin\alpha + \sin\beta$

Return to Review Topic

Suppose $f(x) = \dfrac{x}{2^x}$

Answers:

a) $f(n) = \dfrac{n}{2^n}$

b) $f(n+1) = \dfrac{n+1}{2^{n+1}}$

c) $f(2n) = \dfrac{2n}{2^{2n}}$; This can be simplified into $\dfrac{n}{2^{2n-1}}$

d) $\dfrac{f(n+1)}{f(n)} = \dfrac{n+1}{2^{n+1}} \cdot \dfrac{2^n}{n} = \dfrac{2^n}{2^{n+1}}\left(\dfrac{n+1}{n}\right) = \dfrac{1}{2}\left(\dfrac{n+1}{n}\right)$

Return to Review Topic

Let $f(x) = \dfrac{1}{x-2}$.

a) Evaluate and simplify the difference quotient $\dfrac{f(x+h)-f(x)}{h}$, $h \neq 0$.

b) Take $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$ and compare the result to $f'(x)$.

Answers:

a)

$$\begin{aligned}
\frac{f(x+h)-g(x)}{h} &= \frac{\dfrac{1}{(x+h)-2} - \dfrac{1}{x-2}}{h} \\
&= \frac{(x-2)-(x+h-2)}{(x+h-2)(x-2)} \cdot \frac{1}{h} \\
&= \frac{-h}{(x+h-2)(x-2)(h)} \\
&= \frac{-1}{(x+h-2)(x-2)}
\end{aligned}$$

b)

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \left(\frac{-1}{(x+h-2)(x-2)} \right) \\
&= \frac{-1}{(x-2)^2}
\end{aligned}$$

Compare to:

$$\begin{aligned}
f(x) &= (x-2)^{-1} \\
f'(x) &= -(x-2)^{-2}(1) = \frac{-1}{(x-2)^2}.
\end{aligned}$$

Return to Review Topic