MATH 250 – REVIEW TOPIC 10

Parametric Curves

This section introduces the notion of a parametric curve. Most of the curves you have seen to this point are determined by functions of the form $y = f(x)$. In this context you choose an x, solve for y, and then plot the (x, y) point. This is not quite the case for the example below.

Example 10.1. Graph the parametric curve $x = 2t$, $y = 3t$, $0 \leq t \leq 1$.

In Example 10.1 a third variable, t, is present. It is called the parameter. The idea is to choose a value for t, use this value to determine x and y, and then plot the point (x, y). In a certain sense you never really "see" the variable t, although it very much affects the shape and direction of the curve.

A table for Example 10.1 and the associated graph appear below. In this example, as t increases both x and y increase, and so the direction is as indicated.

t	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	0	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3

Table 10.1.

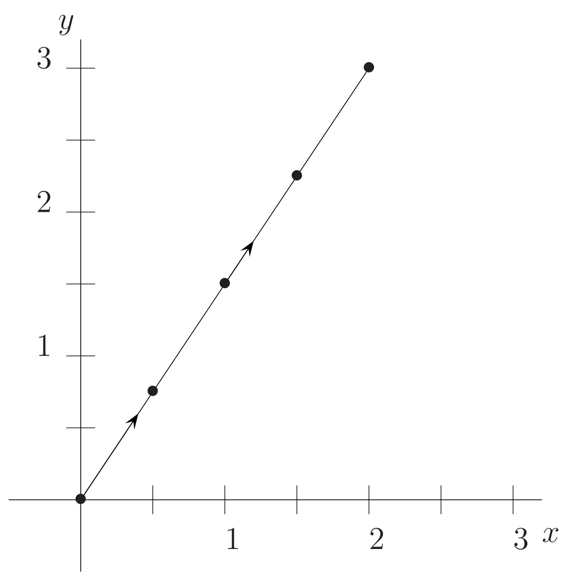


Fig. 10.1.

Let us summarize. A parametric curve in the plane is a curve where the x and y coordinates are determined by a third variable called the parameter. The parameter varies over a certain interval which gives a direction to the curve and leads to values for x and y. Then the (x, y) points are plotted.

Parametric curves are very important in physics and engineering and are needed for sketching curves in three dimensions. In Calc II you will do basic curve sketching and some applications. This topic will be revisited in Calc III much more extensively.

When trying to graph parametric curves, the parameter can sometimes be eliminated so that you have an equation in rectangular form.

In Example 10.1 we can eliminate t as follows.

$$x = 2t \Rightarrow \frac{x}{2} = t; \quad y = 3t \Rightarrow \frac{y}{3} = t$$
$$\Rightarrow \frac{x}{2} = \frac{y}{3} \Rightarrow y = \frac{3}{2}x.$$

We now have to determine the range of x and y.

$$\text{If } t = 0,\ x = 0 \text{ and if } t = 1,\ x = 2 \Rightarrow 0 \le x \le 2$$
$$\text{If } t = 0,\ y = 0 \text{ and if } t = 1,\ y = 3 \Rightarrow 0 \le y \le 3.$$

This means Example 10.1 and Fig. 10.1 cover the same set of points as

$$y = \frac{3}{2}x, \quad 0 \le x \le 2.$$

However, a direction is involved in Example 9.1.

Example 10.2 Graph the parametric curve $x = \cos\theta$, $y = \sin\theta$, $0 \le \theta \le 2\pi$.

In this example the parameter is θ instead of t. We make a table and plot the (x, y) points.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
y	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0

Table 10.2.

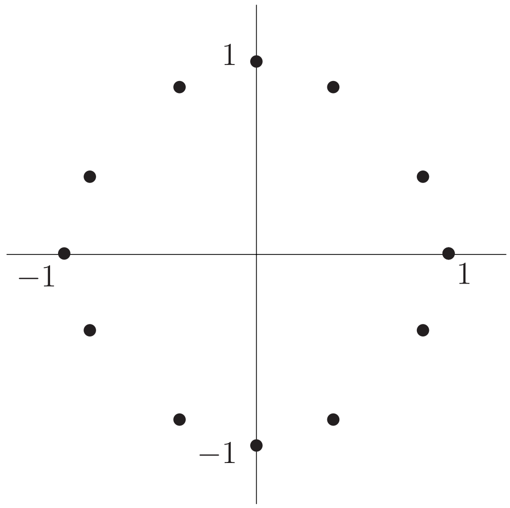


Fig. 10.2.

The shape has the appearance of a circle. This can be verified by eliminating the parameter from the equations in Example 10.2. Specifically,

$$x^2 + y^2 = \cos^2\theta + \sin^2\theta = 1, \quad \text{or}$$
$$x^2 + y^2 = 1 \tag{10.1}$$

We see that Eqn (10.1) and Fig. 10.2 do indeed represent a circle. Table 10.1 indicates that the circle is traversed once in a counterclockwise direction starting at the point $(1, 0)$ as shown in the picture below.

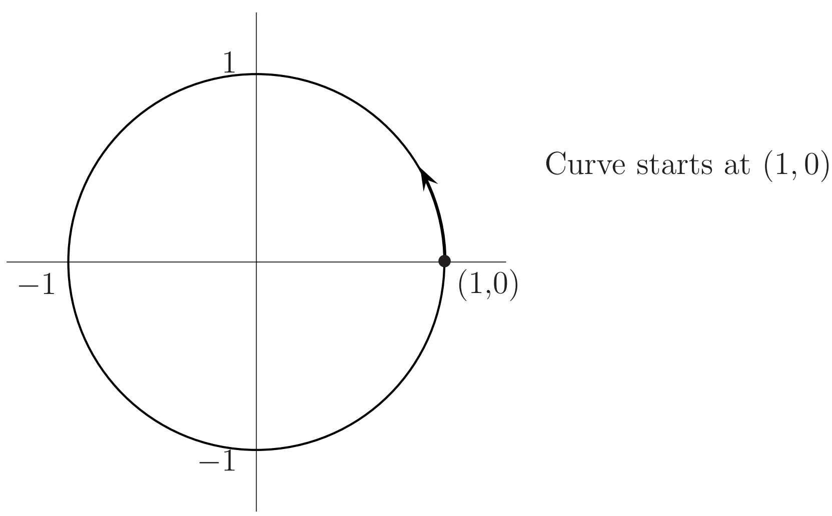


Fig. 10.3.

Remark: If we let $x = \sin\theta$ and $y = \cos\theta$, $0 \le \theta \le 2\pi$, then the curve is a circle starting at $(0, 1)$ which is traversed in a clockwise direction.

Note that Fig. 10.3 does not represent a function since it fails the vertical line test (Math 150, Review Topic 1). However, if we consider the curve in Fig. 10.3 as being determined parametrically by Example 10.1, then every point on the curve is well defined in terms of the parameter θ. This is one of the interesting characteristics of parametric equations. They allow you to define curves and graphs that would be impossible in the "$y = f(x)$" setting.

To further emphasize this point, consider Example 10.2 with parameter θ defined on the interval $0 \leq \theta \leq 4\pi$. What happens now? The circle is traversed twice with each point it passes through being well defined. Again, this could never happen in the $y = f(x)$ format. In Calc II, you will study parametric curves in greater depth.

PRACTICE PROBLEMS for Topic 9

In the following problems, graph the curve and indicate its direction.

10.1. Graph the parametric curve $x = 1 + t$, $y = 1 - t$, $-1 \leq t \leq 1$. Answer

10.2. Graph the parametric curve $x = t$, $y = 2 - t^2$, $0 \leq t \leq 2$. Answer

10.3. Graph the parametric curve $x = \cos\theta$, $y = 1 - \sin^2\theta$, $\theta \geq 0$. Answer

10.4. Graph the parametric curve $x = e^t$, $y = e^{-t}$, $-\infty < t < \infty$. Answer

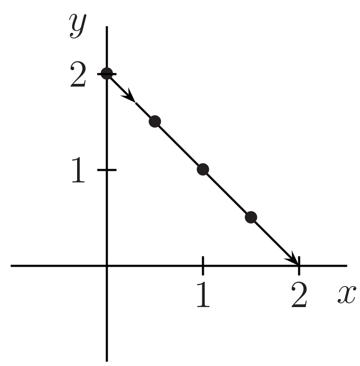
Beginning of Topic Review Topics 250 Skills Assessment

ANSWERS to PRACTICE PROBLEMS (Topic 10 –Parametric Curves)

10.1. Graph $x = 1 + t$, $y = 1 - t$, $-1 \le t \le 1$. Let's make a table.

t	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0

Now plot the pairs (x, y) and note the direction determined by t. That is, the curve starts at $(0, 2)$ (when $t = -1$) and ends at $(2, 0)$ (when $t = 1$).



In this problem the parameter t can be eliminated.

$$\left.\begin{aligned} x = 1 + t \Rightarrow x - 1 = t \\ y = 1 - t \Rightarrow 1 - y = t \end{aligned}\right\} \Rightarrow \begin{aligned} & x - 1 = 1 - y \text{ or} \\ & x + y = 2. \end{aligned}$$

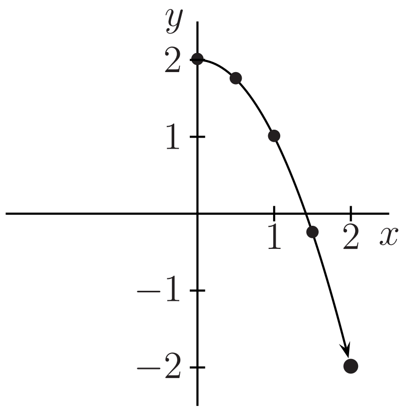
If $t = -1$, $x = 0$ and if $t = 1$, $x = 2 \Rightarrow 0 \le x \le 2$.
If $t = -1$, $y = 2$ and if $t = 1$, $y = 0 \Rightarrow 2 \ge y \ge 0$.

Return to Problem

10.2. Graph $x = t$, $y = 2 - t^2$, $0 \le t \le 2$. Our table is:

t	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
x	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
y	2	$\frac{7}{4}$	1	$-\frac{1}{4}$	-2

Now plot the pairs (x, y) and note the direction determined by t. That is, the curve starts at $(0, 2)$ (when $t = 0$) and ends at $(2, -2)$ (when $t = 2$).



Now eliminate the parameter.

$$x = t \Rightarrow y = 2 - t^2 \Rightarrow y = 2 - x^2 \Rightarrow y = -x^2 + 2.$$

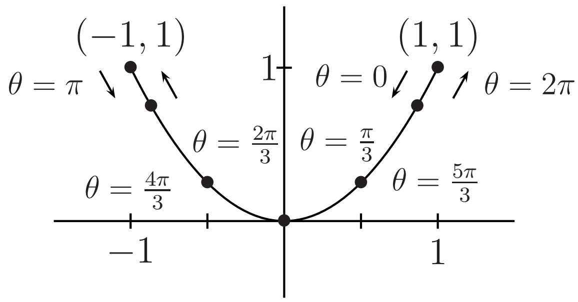
If $t = 0,\ x = 0$ and if $t = 2,\ x = 2 \Rightarrow 0 \le x \le 2.$
If $t = 0,\ y = 2$ and if $t = 2,\ y = -2 \Rightarrow 2 \ge y \ge -2.$

Return to Problem

10.3. Graph $x = \cos\theta$, $y = 1 - \sin^2\theta$, $\theta \geq 0$. First, make a table for $0 \leq \theta \leq 2\pi$.

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
x	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	1
y	1	$\frac{\sqrt{3}}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{4}$	1	$\frac{3}{4}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{3}{4}$	1

Plotting the (x, y) points and noting the direction given by $\theta \geq 0$ yields the following.



To eliminate θ, we write $x^2 = \cos^2\theta = 1 - \sin^2\theta = y$, or $y = x^2$. The problem is more complicated, though, than just tracing the curve of $y = x^2$ in some direction. The curve starts at $(1, 1)$, comes down to $(0, 0)$, up to $(-1, 1)$ and returns back to $(1, 1)$. This is for $0 \leq \theta \leq 2\pi$. Since $\cos\theta$ and $\sin\theta$ are periodic of period 2π, as θ assumes values $\geq 2\pi$ the above points are repeated. So, the curve starts at $(1, 1)$ and travels back and forth from $(1, 1)$ to $(-1, 1)$ as θ varies.

Thus we always have $-1 \leq x \leq 1$ and $0 \leq y \leq 1$, which means the curve "swings back and forth" over the bottom part of the parabola $y = x^2$.

Return to Problem

10.4. Graph $x = e^t$, $y = e^{-t}$, $-\infty < t < \infty$. First we make a table.

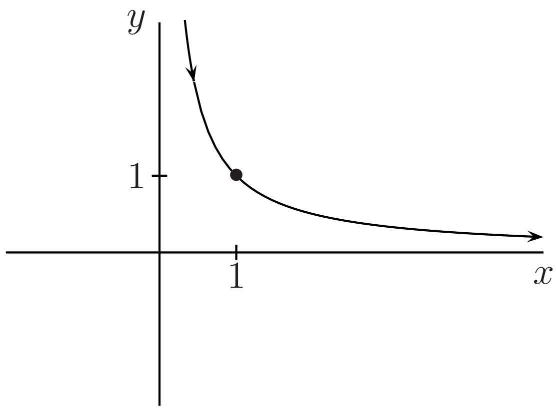
t	0	1	2	-1	-2
x	1	e	e^2	e^{-1}	e^{-2}
y	1	e^{-1}	e^{-2}	e	e^2

etc.

Notice that x and y are <u>always</u> positive. Also, $\lim_{t\to\infty} x = \infty$, $\lim_{t\to\infty} y = 0$, $\lim_{t\to-\infty} x = 0$, and $\lim_{t\to-\infty} y = \infty$. Before plotting the above points, let us eliminate the parameter.

$$y = e^{-t} = \frac{1}{e^t} = \frac{1}{x} \text{ or}$$
$$y = \frac{1}{x}\,.$$

Since x and y must always be positive, the graph is in the first quadrant with the direction as indicated.



Return to Problem