MATH 250 – TOPIC 13
INTEGRATION

A. Integration of Common Functions

There are two types of integrals, indefinite and definite.

$$\text{An indefinite integral is written as } \int f(x)\,dx,$$

and

$$\text{a definite integral is written as } \int_a^b f(x)\,dx.$$

Evaluating an indefinite integral requires finding a function $F(x)$ whose derivative is the integrand $f(x)$. That is,

$$\int f(x)\,dx = F(x) \text{ where } F'(x) = f(x).$$

F(x) is called an antiderivative of f(x). For example

$$\int 2x\,dx = x^2 + c \text{ (c representing a constant) since } \frac{d}{dx}(x^2+c) = 2x.$$

Similarly

$$\int \cos x\,dx = \sin x + c \quad \text{since} \quad \frac{d}{dx}(\sin x + c) = \cos x.$$

The process for evaluating a definite integral is slightly different. First, find an antiderivative. Then evaluate the antiderivative at its limits (a and b) and subtract the result. Thus,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Example:

$$\int_1^3 2x\,dx = (x^2+c)\Big|_1^3 = (3^2+c) - (1^2+c)$$
$$3^2 - 1^2 = 8.$$

Note: It is customary to leave out the c with definite integrals because the c always cancels itself out in the subtraction.

Exercise A.1: Find the antiderivatives of the following basic functions. What is interesting is that in Calculus II you will cover five-six different methods of integration. In the end, however, all of the methods reduce to evaluating one of these basic integrals.

$\int f(x)\,dx$	F
$\int x^n\,dx,\ n \neq 1$	
$\int x^{-1}\,dx = \int \frac{dx}{x}$	
$\int e^x\,dx$	
$\int \sin x\,dx$	
$\int \cos x\,dx$	
$\int \sec^2 x\,dx$	
$\int \sec x \tan x\,dx$	
$\int \csc^2 x\,dx$	
$\int \csc x \cot x\,dx$	

Answers

Rewriting Integrals to Fit Basic Forms

Sometimes an integral must be rewritten before it can be integrated. For example,

$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx = -x^{-1} + c \quad \text{or} \quad -\frac{1}{x} + c$$

or

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + c$$

or

$$\int \frac{e^{2x}}{e^x}\,dx = \int e^x\,dx = e^x + c.$$

or

$$\int (1 + \tan^2 x)\,dx = \int \sec^2 x\,dx = \tan x + c.$$

Each integrand was rewritten in one of the basic forms covered in Exercise A.1. This is our first integration strategy. **Always try to rewrite an integral in one of the basic forms.** (Additional strategies will be discussed later.)

Practice Problems

13A.1. $\displaystyle\int x\sqrt{x}\,dx$ Answer

13A.2. $\displaystyle\int_{\pi/4}^{\pi/2} \cos\theta\,d\theta$ Answer

13A.3. $\displaystyle\int \frac{dx}{e^{-x}}$ Answer

13A.4. $\displaystyle\int_{-1}^{1} t^3 dt$ Answer

B. Constant, Sum, and Difference Rules

We have a few basic rules for integration that are similar to the constant, sum, and difference rules for differentiation. Unfortunately though, there is no product, quotient or chain rule analog for integration. The first rule is the constant rule. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$. This means you can ignore the multiplicative constant when finding an antiderivative.

Examples:

$$\begin{aligned}\int 3\sin x\,dx = 3\int \sin x\,dx &= 3(-\cos x) + c \\ &= -3\cos x + c\end{aligned}$$

$$\begin{aligned}\int_0^8 \pi x^{2/3}\,dx = \pi\int_0^8 x^{2/3}\,dx &= \pi\frac{x^{1+2/3}}{\frac{2}{3}+1}\Bigg|_0^8 \\ &= \pi\frac{3}{5}x^{5/3}\Bigg|_0^8 = \frac{3\pi}{5}[8^{5/3} - 0^{5/3}] = \frac{96\pi}{5}\end{aligned}$$

Exercises: Can you find

B.1. $\displaystyle\int -2\sec^2 x\,dx,$ Answer

B.2. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{4}{\sin^2 x}\,dx,$ Answer

B.3. $\displaystyle\int e^{x+1}\,dx?$ Hint: $e^{x+1} = e^x \cdot e$ Answer

Another important rule involves the integral of the sum or difference of two functions.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Here you find the antiderivatives of the individual functions and add or subtract the result.

Examples:

1. $$\int_1^2 (x^{-1/2} + e^x)\,dx = \int_1^2 x^{-1/2}\,dx + \int_1^2 e^x\,dx$$
$$= 2x^{1/2}\Big|_1^2 + e^x\Big|_1^2$$
$$= 2[2^{1/2} - 1^{1/2}] + [e^2 - e]$$
$$= 2\sqrt{2} - 2 + e^2 - e$$

2. $$\int \left(\frac{1}{x} + \tan x\right) = \int \frac{dx}{x} + \int \tan x\,dx$$
$$= \ln|x| + -\ln|\cos x| + c$$

3. $$\int \sqrt{x}(x^2-2)dx = \int x^{5/2}dx - 2\int x^{1/2}dx$$
$$= \frac{2}{7}x^{7/2} - \frac{4}{3}x^{3/2} + c$$

Practice Problems. Can you find the following?

13B.1. $\displaystyle\int_{\pi/6}^{\pi/4} (\cos x \tan x + 1)\,dx$ Answer

13B.2. $\displaystyle\int \left(\frac{x^3+x^2+1}{x}\right)\,dx$ Answer

13B.3. $\displaystyle\int_{\pi/6}^{\pi/3} (\cot^2 x + 1)\,dx$ Answer

13B.4. $\displaystyle\int \tan^2 x\,dx$ Answer

13B.5. $\displaystyle\int (e^x - \cos x)\,dx$ Answer

13B.6. $\displaystyle\int (x^2-1)^2\,dx$ Answer

13B.7. $\displaystyle\int \frac{e^{2x} - e^x}{e^x}\,dx$ Answer

C. Substitution

Why Substitution Works

You've already seen how to use the method of substitution to evaluate integrals. This method of integration "compensates" for the chain rule in differentiation. Let's start our review of substitution by looking at some derivatives involving the chain rule. (Review Topic 12E).

1. $\dfrac{d}{dx}e^{x^2+1} = e^{x^2+1}(2x)$
2. $\dfrac{d}{dx}\sin(\ln x) = \cos(\ln x)\cdot\dfrac{1}{x}$
3. $\dfrac{d}{dx}(x^3+3x-1)^{1/2} = \dfrac{1}{2}(x^3+3x-1)^{-1/2}(3x^2+3)$

Since integration can be considered as the "reverse" of differentiation, if you were asked to evaluate

$$\int e^{x^2+1}(2x)\,dx,$$

you can see from above the answer is $e^{x^2+1}+c$. Similarly, 2. and 3. above imply that

$$\int \cos(\ln x)\frac{dx}{x} = \sin(\ln x) + c,$$

and

$$\int \frac{1}{2}(x^3+3x-1)^{-1/2}(3x^2+3)\,dx = (x^3+3x-1)^{1/2} + c.$$

If we didn't know the answer though, how could we work these problems? We make a substitution.

In algebra you probably made substitutions to simplify expressions. For example, solving the following equation

$$x^{-2/3} - x^{-1/3} - 2 = 0$$

appears pretty hard. If we let $u = x^{-1/3}$ though, the problem becomes

$$u^2 - u - 2 = 0;$$

Substitution helped us recognize that the equation was quadratic and can be solved. This same idea of substitution applies to integration.

Method of Substitution

Let's go back to $\displaystyle\int e^{x^2+1}(2x)\,dx$.

Suppose $u(x) = x^2 + 1$. Then $\dfrac{du}{dx} = 2x \Rightarrow du = 2x\,dx$. This means

$$\begin{aligned}\int \overbrace{e^{x^2+1}}^{u}\,\overbrace{2x\,dx}^{du} &= \int e^u du \\ &= e^u + c = e^{x^2+1} + c.\end{aligned}$$

WARNING! When using substitution it is important that your substitution accounts for all of the x-terms and dx in the starting integral. The new integral should only be in terms of u and du (no x's!).

Now try $\displaystyle\int \cos(\ln x)\frac{dx}{x}$.

Let $u(x) = \ln x$. Then $du = \dfrac{dx}{x}$ and you end up integrating

$$\int \cos u\,du = \sin u + c = \sin(\ln|x|) + c.$$

Exercise C.1. What substitution do you think you would use with

$$\int \frac{1}{2}(x^3 + 3x - 1)^{-1/2}(3x^2 + 3)\,dx?$$

Answer

The method involved with substitution is:

1. Choose a substitution $u = f(x)$.
2. Compute $\dfrac{du}{dx}$.
3. Rewrite integral in terms of u and du. (No x-terms left)
4. Evaluate integral in terms of u.
5. Replace u by $f(x)$.

The result is the antiderivative in terms of x.

Note: As integrals get more complicated, we encourage you to check them by differentiating. This also allows you to practice derivatives.

Each of the following examples has $e^{f(x)}$ in the integral. Let $u(x) = f(x)$, determine du, substitute and integrate. Don't forget to put the final answer in terms of x!

Exercises:

C.2. $\displaystyle\int e^{\sin x} \cos x \, dx$ Answer

C.3. $\displaystyle\int e^{2x^{1/2}} \frac{dx}{x^{1/2}}$ Answer

C.4. $\displaystyle\int e^{3x^2+2x-1}(6x+2)\, dx$ Answer

Now consider $\displaystyle\int e^{x^2+1} x\, dx$. Again we let $u = x^2 + 1$. But now $du = 2x\, dx$ and we don't have the 2 in the integral. Recall $\displaystyle\int cf(x)\, dx = c \int f(x)\, dx$.

So, solving for $x\,dx = \frac{1}{2}du$ we have

$$\int e^{x^2+1}x\,dx = \int e^u \frac{1}{2}\,du = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c$$
$$= \frac{1}{2}e^{x^2+1} + c$$

Note: Since the integral contained "only" $x\,dx$ instead of $2x\,dx$, wouldn't you expect the final answer to be $\frac{1}{2}$ as much.

Each of the following examples involves a trigonometric function with an argument $f(x)$. i.e., $\sin(f(x))$, $\tan(f(x))$, etc. Again let $u = f(x)$, determine du, substitute, and integrate.

Exercises:

C.5. $\displaystyle\int x\,\sin(x^2+1)\,dx$ Answer

C.6. $\displaystyle\int (e^x+1)\,\sec^2(e^x+x)\,dx$ Answer

C.7. $\displaystyle\int (x^2+1)\,\cos(x^3+3x-1)\,dx$ Answer

Consider a more complicated example. Evaluate

$$\int x\sqrt{x+1}\,dx.$$

Let $u = x+1$. Then $du = dx$ and $x = u-1$. (Remember, all x-terms must change to u-terms!) Thus

$$\int x\sqrt{x+1}dx = \int (u-1)u^{1/2}du$$
$$= \int (u^{3/2} - u^{1/2})du.$$

Since everything to do with x was changed to u, this means you substituted correctly.

Exercise C.8. Finish the preceding problem. Answer

How can we recognize when substitution won't help?

Consider the problem $\int x^2 e^{x^2+1}\,dx$, when $x > 0$. If we let $u = x^2+1$, then $du = 2x\,dx$ or $\dfrac{du}{2} = x\,dx$.

$$\Rightarrow \int x^2 e^{x^2+1}\,dx = \int x \cdot \overbrace{e^{x^2+1}}^{e^u} \cdot \overbrace{x\,dx}^{\frac{1}{2}du}$$

To get rid of x, we write $u - 1 = x^2$, or $x = \sqrt{u-1}$. (We take $\sqrt{u-1}$ instead of $-\sqrt{u-1}$ since $x > 0$.). This means

$$\int x^2 e^{x^2+1}\,dx = \frac{1}{2}\int \sqrt{u-1}\,e^u du.$$

We have substituted correctly in the sense that all x terms have been changed to u. However, the new integral is just as complicated as the old one. The <u>main</u> <u>goal</u> of substitution is to change the starting integral into another one that you can actually integrate. In this case, our substitution $u = x^2+1$ did not help us. We must either try a different substitution or use another method of integration (that you will learn in Calculus II).

Substitution also effects the limits of integration.

An integral can't be written in terms of u with limits intended for x. We must use the substitution to change the "x-limits" to "u-limits".

Example: Evaluate $\displaystyle\int_1^3 e^{x^2+1} 2x\,dx$.

Let $u = x^2+1$. Then

$$\begin{aligned} &\text{if } x = 1, \qquad u = 1^2 + 1 = 2, \quad \text{and} \\ &\text{if } x = 3, \qquad u = 3^2 + 1 = 10. \end{aligned}$$

Thus

$$\int_1^3 e^{x^2+1} 2x\,dx = \int_2^{10} e^u du = e^u\big|_2^{10} = e^{10} - e^2.$$

How do you know what to substitute?

Suppose your integrand has a term of the form:

$$e^{f(x)}, \qquad \frac{1}{f(x)}\,dx, \qquad \operatorname{trig}(f(x)), \qquad (f(x))^p.$$

A general rule of thumb is let $u = f(x)$, determine du, and get rid of all the x's and dx.

As a final note, observe that $\displaystyle\int \tan x\,dx$ can be evaluated by a substitution.

$$\int \tan x\,dx = \int \frac{\sin x\,dx}{\cos x}.$$

Let $u = \cos x \Rightarrow du = -\sin x\,dx$

$$\begin{aligned} &= \int -\frac{du}{u} = -\ln|u| + c \\ &\qquad\qquad \left.\begin{aligned} &= -\ln|\cos x| + c \\ &= \ln|\sec x| + c \end{aligned}\right\} \text{These are both correct forms. Why are they equivalent?} \end{aligned}$$

Practice Problems. Evaluate the following. Change limits when necessary. Not all problems require substitution.

13C.1. $\displaystyle\int \frac{dx}{\sqrt{x-1}}$ Answer

13C.2. $\displaystyle\int_2^3 \frac{x\,dx}{\sqrt{x-1}}$ Answer

13C.3. $\displaystyle\int x\sqrt{x^2-1}\,dx$ Answer

13C.4. $\displaystyle\int \sqrt{x}(x-1)\,dx$ Answer

13C.5. $\displaystyle\int_0^{\pi/2} \cos\left(\frac{2x}{3}\right)\,dx$ Answer

13C.6. $\displaystyle\int \frac{e^x\,dx}{e^x+1}$ Answer

13C.7. $\displaystyle\int_0^{\pi/6} \tan^2(2x)\,dx$ Answer

To finish up this section, let's try to make your life a little easier with some integrations you should be able to do quickly and (hopefully) without substitution. For example $\displaystyle\int e^{3x}\,dx$, $\displaystyle\int \sin 2x\,dx$, $\displaystyle\int \frac{dx}{x-1}$. These integrals are of the form

$$\int e^{ax}\,dx, \qquad \int \mathrm{trig}(ax), \qquad \int \frac{dx}{x+k}$$

The substitution for the first two is $u = ax$ and $du = a\,dx$. This leads to $\frac{1}{a}\int \underline{\hspace{2em}}\,du$. For the last integral, let $u = x + k$ which gives a result $\ln|x+k| + c$.

Exercise C.9. Without making a formal substitution, try completing the following table.

	a =	F(x) =
$\int \sin 2x \, dx$		$-\frac{1}{2}\cos 2x + c$
$\int e^{3x} \, dx$		
$\int e^{-x/2} \, dx$		
$\int \sec 3x \tan 3x \, dx$		
$\int \cos \pi x \, dx$		
$\int \frac{dx}{x+2} \, dx$		
$\int \frac{dx}{2-x} \, dx$		

Answer

D. Area Under a Curve

One of the first applications covered in your previous study of definite integrals was finding the area under a curve. Actually, the integral $\int_a^b f(x)\,dx$ can be defined as the area between the curve $y = f(x)$ and the x-axis for $a \le x \le b$.

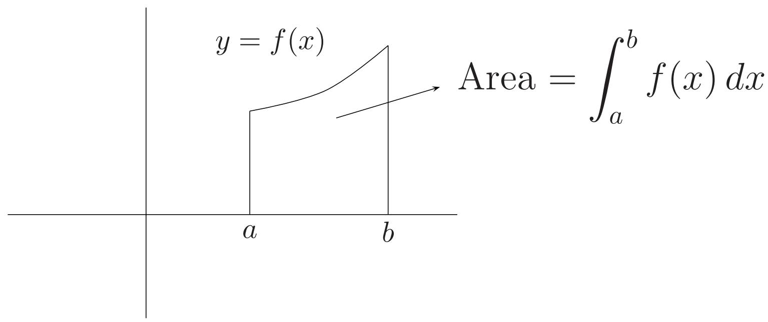


Fig. 13D.1.

As you (hopefully!) remember, we first evaluated $\int_a^b f(x)\,dx$ by setting it equal to the limit of Riemann sums. The idea is as follows.

Arguing intuitively, we formed rectangles as below.

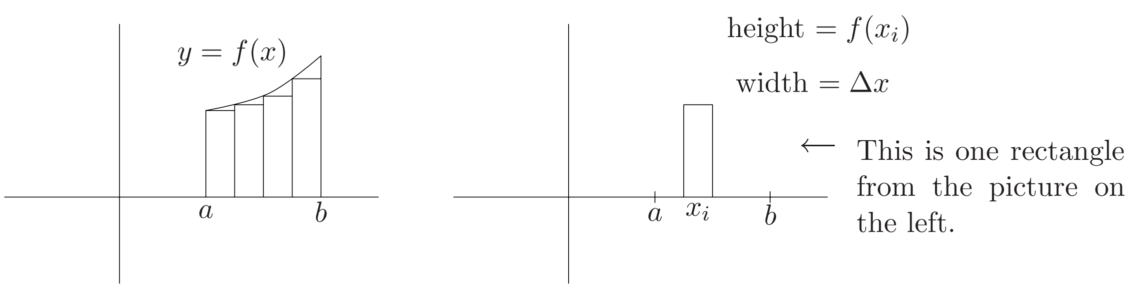


Fig. 13D.2.

The exact area, i.e. the integral, is approximated by adding the area of the rectangles (in this example there are four rectangles). The area of each rectangle = (height)(width). The height $= y = f(x_i)$, where x_i is some x-value in the base of the rectangle, and the width $= \Delta x$. For the drawing on the left in Fig. 13D.2, we write

$$\text{Area } = \int_a^b f(x)\,dx = \text{(approximately) } \sum_{i=1}^{4} f(x_i)\Delta x.$$

The expression on the right in the line above is called a Riemann sum with $n = 4$. Instead of forming a Riemann sum with 4 rectangles, we could have used 6, 10, 6000 rectangles, etc. The more rectangles we form, the better our approximation will be. If we take the limit as $n \to \infty$, where n is the number of rectangles, we obtain the exact area.

Intuitively then, the integral $\displaystyle\int_a^b f(x)\,dx$ can be thought of as the sum of an infinite number of rectangles. The integral sign represents the sum, "$f(x)$" represents the height of the rectangle, and "dx" corresponds to the width of the rectangle. The "dx" also indicates that the variable of integration or independent variable is x.

Fortunately, the Fundamental Theorem of Calculus allows us to find the area (evaluate the integral) by using antiderivatives instead of trying to add an infinite number of rectangles.

Example 13D.1. Find the area under $f(x) = \sin x$ and the x-axis from 0 to π.

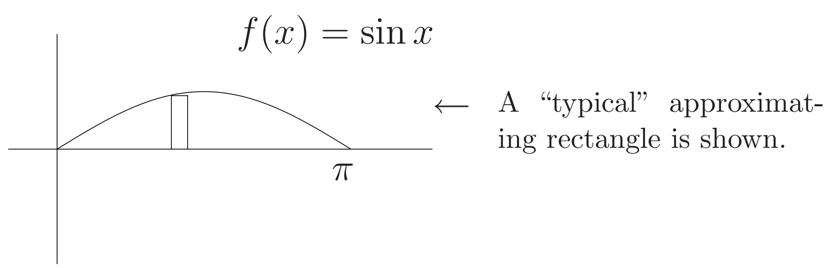


Fig. 13D.3.

$$\begin{aligned}\text{Area } &= \text{(intuitively) "the sum of the rectangles from } x = 0 \text{ to } x = \pi\text{"}\\ &= \int_0^\pi \overbrace{\sin x}^{\text{"ht"}}\ \overbrace{dx}^{\text{"width"}} = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) + 1 = 2.\end{aligned}$$

What happens if the curve dips below the x-axis? For example, consider

$f(x) = \sin x,\ -\dfrac{\pi}{2} \le x \le \pi.$

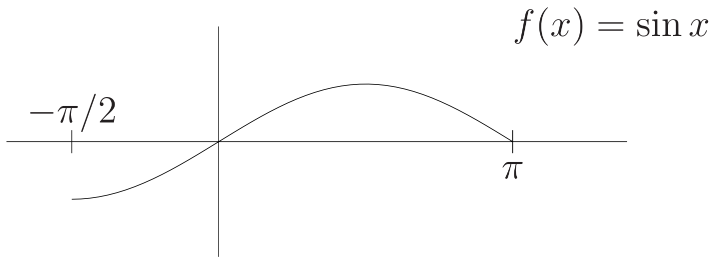


Fig. 13D.4.

First, notice that on the interval $-\dfrac{\pi}{2} \le x \le 0$,

$$\int_{-\pi/2}^{0} f(x)\,dx = \int_{-\pi/2}^{0} \sin x\,dx = -\cos x\Big|_{-\pi/2}^{0} = -1.$$

A negative answer is not surprising since $f(x_i)$, the "height" of an approximating rectangle, is negative when $-\dfrac{\pi}{2} \le x \le 0$. Thus, to find the area enclosed by $f(x) = \sin x$, $-\dfrac{\pi}{2} \le x \le \pi$, we must:

(i) split the x interval into subintervals where $f(x) \ge 0$ or where $f(x) \le 0$;

(ii) integrate $f(x)$ over each subinterval;

(iii) take the absolute value of each resulting integral.

In Fig. 13D.4, the area is:

$$\begin{aligned}\text{Area} &= \left|\int_{-\pi/2}^{0} \sin x\,dx\right| + \int_{0}^{\pi} \sin x\,dx \\ &= |-1| + 2 = 3.\end{aligned}$$

Warning: If you are asked simply to evaluate an integral, then just evaluate it and don't worry about whether $f(x)$ is positive or negative. For example,

$$\int_{-\pi/2}^{\pi} \sin x\,dx = -\cos x\Big|_{-\pi/2}^{\pi} = -(-1) - (0) = 1.$$

Notice from Fig. 13D.4 that the integration process yields the "net" value. That is, the negative part, $\int_{-\pi/2}^{0} \sin x\,dx$, cancels $\int_{0}^{\pi/2} \sin x\,dx$, which is positive, so that we only have left $\int_{\pi/2}^{\pi} \sin x\,dx$. Bottom line, if you are asked to evaluate an integral, just plunge ahead. (That's why in problem 13A.4, $\int_{-1}^{1} t^3\,dt = 0$.) If you are asked to find the area, you must worry about whether $f(x) \geq 0$ or ≤ 0 and proceed accordingly.

It is easy to extend these ideas to finding the area between two curves.

Example 13D.2. Find the area bounded by the curves $f(x) = x^2$ and $g(x) = 2x$.

First, determine where the curves intersect. Do this by setting $f(x) = g(x)$. Thus, $x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \Rightarrow$ the curves intersect at $x = 0$ and $x = 2$ as indicated below.

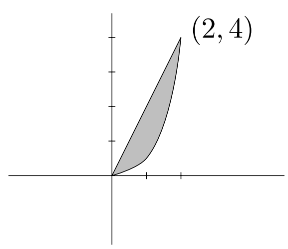


Fig. 13D.5.

You can use two approaches to find the desired area. First, find the area under the upper curve $g(x) = 2x$ from $x = 0$ to $x = 2$ and then subtract the area under the lower curve $f(x) = x^2$ from $x = 0$ to $x = 2$.

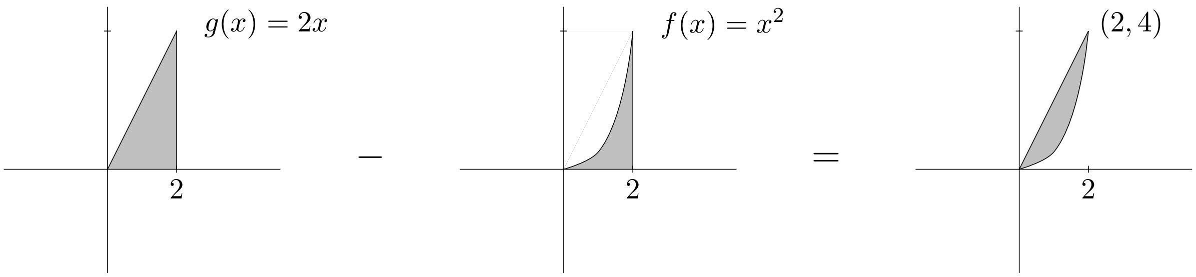
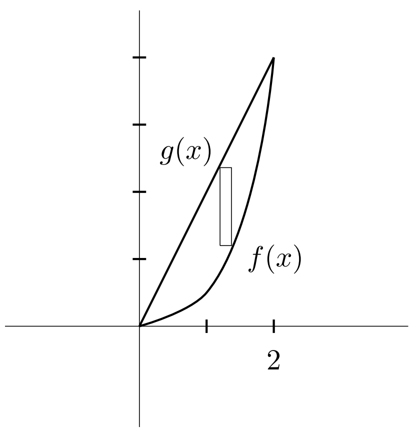


Fig. 13D.6.

Using integrals to represent the above areas, we have

$$\int_0^2 2x\,dx - \int_0^2 x^2\,dx = x^2\Big|_0^2 - \frac{x^3}{3}\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}\,.$$

For the second approach, using Fig. 13D.5 you can draw one of the rectangles that would approximate the area and then write the corresponding integral.



Height of rectangle	$= g(x_i) - f(x_i)$ $= 2x_i - x_i^2$
Width of rectangle	$= dx$
Area of rectangle	$= (2x_i - x_i^2)dx$

Fig. 13D.7.

The rectangles are formed when $0 \le x \le 2$. Thus,

$$\text{area} = \int_0^2 [g(x) - f(x)]\,dx = \int_0^2 (2x - x^2)\,dx = x^2 - \frac{x^3}{3}\Big|_0^2 = \frac{4}{3}\,,$$

which, of course, gives the same answer as the first approach.

A little hint: Notice how much it helps to sketch the functions no matter which approach you use.

Another hint: In the second approach above, you are finding the height of the rectangle by saying the height = "upper curve − lower curve". This idea of "upper curve − lower curve" works no matter where the curves are. That is, the curves could both be below the x-axis, one curve could be above and the other below, or both could be above the x-axis like Fig. 13D.7. It doesn't matter. The height of the rectangle is always the "upper curve − lower curve" and is always positive. Consider the following example.

Example 13D.3. Find the area bounded by the curves $f(x) = x$ and $g(x) = x^3$.

Setting $f(x) = g(x)$ we see that $x = x^3 \Rightarrow x^3 - x = 0 \Rightarrow x(x^2 - 1) = 0 \Rightarrow$ the curves intersect at $x = -1, 0, 1$.

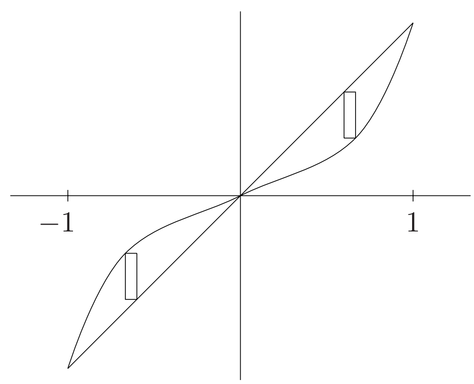


Fig. 13D.8.

Notice that for $-1 < x < 0$, $g(x) = x^3$ is above $f(x) = x$. However, for $0 < x < 1$, $f(x) = x$ is above $g(x) = x^3$. This means we should divide the problem into two parts: finding the area when $-1 < x < 0$ and finding the area when $0 < x < 1$. Looking at Fig. 13D.8 and remembering that the height of an approximating rectangle is always the "upper curve − lower curve", we see that the area when $-1 < x < 0$ is given by:

$$\int_{-1}^{0} (x^3 - x)\, dx = \frac{x^4}{4} - \frac{x^2}{2}\bigg|_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4}\,.$$

For $0 < x < 1$, the area is:

$$\int_{0}^{1} (x - x^3)\, dx = \frac{x^2}{2} - \frac{x^4}{4}\bigg|_{0}^{1} = \left(\frac{1}{2} - \frac{1}{4}\right) - 0 = \frac{1}{4}\,.$$

Adding the two results: $\dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2}$.

Of course, we could have used symmetry and said that the area is $2\left[\displaystyle\int_0^1 (x - x^3)\,dx\right]$ to arrive at $\dfrac{1}{2}$.

There will be times in Calc II when it will be necessary to consider integrals where y is the independent variable and x is the dependent variable.

Example 13D.4. Find the area in the first quadrant bounded by $x = y^2$, the y-axis, and $y = 1$.

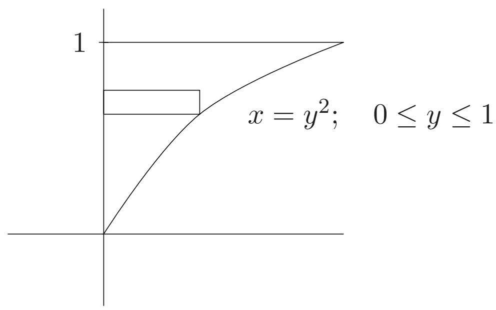


Fig. 13D.9.

Now our approximating rectangles are horizontal. The “height or length” is some function $x = h(y)$, and the “width” is “dy”. In Fig. 13D.9, the area of an approximating rectangle is (length)(width) = “$y^2\,dy$”. Now “add up” all the rectangles by integrating. Since dy varies from $y = 0$ to $y = 1$,

$$\text{area} = \int_0^1 y^2\,dy = \left.\frac{y^3}{3}\right|_0^1 = \frac{1}{3}\,.$$

We offer one final example to you.

Example 13D.5. Find the area bounded by the curves $y = \dfrac{12}{x}$, $y = \dfrac{3}{2}\sqrt{x}$, and $y = \dfrac{x}{3}$.

This problem is definitely more challenging. First, draw a picture.

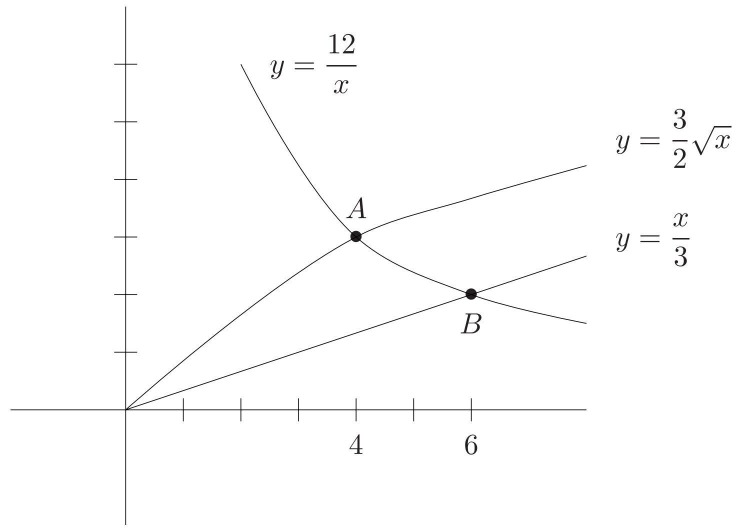


Fig. 13D.10.

To find the point of intersection A, set $\dfrac{12}{x} = \dfrac{3}{2}\sqrt{x} \Rightarrow 8 = x\sqrt{x} = x^{3/2} \Rightarrow$ $x = 4$, $y = 3$. For B, set $\dfrac{12}{x} = \dfrac{x}{3} \Rightarrow x = 6$, $y = 2$.

Now draw an arbitrary rectangle with width "dx". In trying to do this, we see that there are two possible rectangles which can be described, as below.

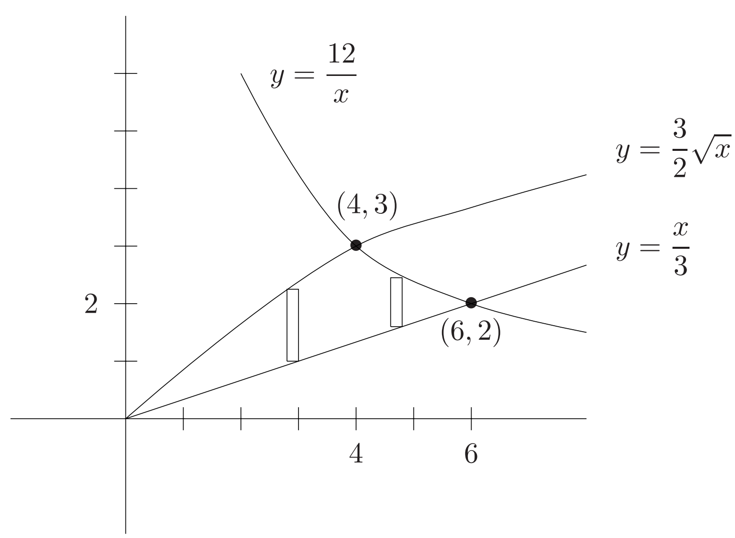


Fig. 13D.11.

The rectangle on the left has height "upper curve-lower curve" $= \left(\frac{3}{2}\sqrt{x} - \frac{x}{3}\right)$, for $0 \leq x \leq 4$ and the rectangle on the right has height $= \left(\frac{12}{x} - \frac{x}{3}\right)$ for $4 \leq x \leq 6$. Since we have two different descriptions, we need two integrals.

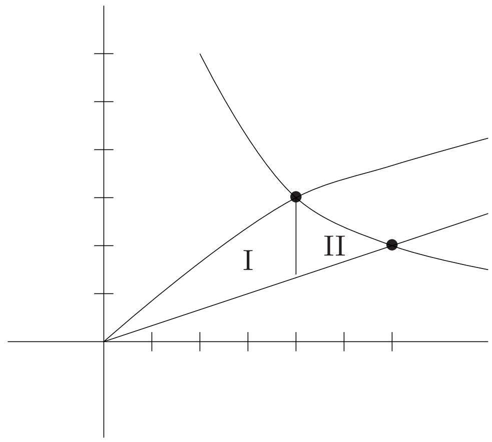


Fig. 13D.12.

$$\begin{aligned}
\text{Area} = \text{I} + \text{II} &= \int_0^4 \left(\frac{3}{2}\sqrt{x} - \frac{x}{3}\right) dx + \int_4^6 \left(\frac{12}{x} - \frac{x}{3}\right) dx \\
&= \left(\frac{3}{2}\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^2}{6}\right)\Bigg|_0^4 + \left(12\ln x - \frac{x^2}{6}\right)\Bigg|_4^6 \\
&= \left(8 - \frac{8}{3}\right) - 0 + (12\ln 6 - 6) - \left(12\ln 4 - \frac{8}{3}\right) \\
&= 2 + 12\ln\left(\frac{3}{2}\right).
\end{aligned}$$

Exercise D.1. Find the area enclosed by the three curves in Example 13D.5 using horizontal rectangles. Answer

Practice Problems. Find the area of the regions bounded by the given curves.

13D.1. $f(x) = \cos x$, x-axis, $x = \frac{\pi}{6}$, and $x = \frac{\pi}{3}$ Answer

13D.2. $f(x) = \dfrac{1}{x^2},\ g(x) = x$ and $x = e^2$ Answer

13D.3. $f(x) = \cos x,\ g(x) = \sin x,\ 0 \leq x \leq 2\pi$ Answer

13D.4. Area of the region to the right of the y-axis and bounded by $x = 2y - y^2$.

Answer

E. Integration Summary

If you feel that integration is not as straightforward as differentiation, you are correct. Complex processes (like limits and integration) are always more difficult to master. Here are a few guidelines that can help when evaluating integrals.

1) Memorize the basic list of integration formulas (Review Topic 13A, Exercise A.1); this includes power, exponential, log, and trigonometric rules.

2) Try to make a substitution $u = g(x)$ that enables the integrand to transform to a basic formula. Learning what to choose for "u" comes after much time and practice.

3) If u substitution doesn't apply, try to alter the integrand. This may involve trig identities or various algebraic manipulations (see Exercise E.1 below). Imagination helps but good algebra/trig skills help more.

4) CHECK more answers to integrals by differentiating (rather than by looking in the back of the book). The more you analyze differentiation, the better you get at integration.

A real challenge in integration involves evaluating integrals that are similar in appearance yet require different methods. Consider the following example.

Example 1: Evaluate the following:

a) $\displaystyle\int \sqrt{x}(x^2-1)\,dx$ b) $\displaystyle\int x\sqrt{x^2-1}\,dx$ c) $\displaystyle\int x^2\sqrt{x-1}\,dx$

Solution:

a) Expand, then integrate using the power rule.

$$\int \sqrt{x}(x^2-1)\,dx = \int (x^{5/2}-x^{1/2})\,dx = \frac{2}{7}x^{7/2} - \frac{2}{3}x^{3/2} + c$$

b) Substitution. Let $u = x^2-1$.

$$\int x\sqrt{x^2-1}\,dx = \frac{1}{2}\int u^{1/2}du = \frac{1}{3}(x^2-1)^{3/2} + c$$

c) Substitution. Let $u = x-1$.

$$\begin{aligned}\int x^2\sqrt{x-1}\,dx &= \int \overbrace{x^2}^{(u+1)^2}\,\overbrace{\sqrt{x-1}}^{u^{1/2}}\,\overbrace{dx}^{du}\\ &= \int (u+1)^2u^{1/2}du\\ &= \int (u^{5/2}+2u^{3/2}+u^{1/2})\,du\\ &= \frac{2}{7}(x-1)^{7/2} + \frac{4}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + c\end{aligned}$$

In any substitution, you must acount for the entire integrand.

Now it's time to see if you are ready to begin Calc II.

Exercise E.1: Evaluate each set of integrals.

I. $\displaystyle\int \frac{x-1}{x^2}\,dx$ and $\displaystyle\int \frac{x^2}{x-1}\,dx$ Answer

II. $\displaystyle\int x\,e^{x^2}dx$ and $\displaystyle\int x\,e^{2x}dx$ Answer

III. $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ and $\displaystyle\int \frac{2}{1+e^x}\,dx$ Answer

IV. $\displaystyle\int \frac{\cos x}{1+\sin x}\,dx$ and $\displaystyle\int \frac{\cos^2 x}{1+\sin x}\,dx$ Answer

Conclusion

In Calc II you will learn more ways to evaluate an integral. Regardless of the method, the intent (strategy) is always the same; manipulate and find a form that can be integrated.

We've done all we can. The rest is up to you. Practice doesn't make one perfect, but it sure helps. Good luck in Calc II.

Beginning of Topic Review Topics 250 Skills Assessment

Answers:

$\int f(x)\,dx$	$F(x)$
$\int x^n\,dx$	$\frac{x^{n+1}}{n+1}+c,\ n\neq -1$
$\int x^{-1}\,dx = \int \frac{dx}{x}$	$\ln\lvert x\rvert + c$
$\int e^x\,dx$	$e^x + c$
$\int \sin x\,dx$	$-\cos x + c$
$\int \cos x\,dx$	$\sin x + c$
$\int \sec^2 x\,dx$	$\tan x + c$
$\int \sec x \tan x\,dx$	$\sec x + c$
$\int \csc^2 x\,dx$	$-\cot x + c$
$\int \csc x \cot x\,dx$	$-\csc x + c$

Question: Why is c included in the antiderivative of an indefinite integral?

a) My instructor told me to put it there.

b) c stands for correct. I want my instructor to know I've checked my answer.

c) Since the derivative of any constant is 0, the inclusion of c suggests the antiderivative is not a single function but rather a family of functions that differ only by a constant.

Return to Review Topic

13A.1. $\int x\sqrt{x}\,dx = \int x\,x^{1/2}\,dx = \int x^{3/2}\,dx = \dfrac{x^{5/2}}{\frac{5}{2}} + c = \dfrac{2}{5}x^{5/2} + c$

Return to Problem

13A.2. $\int_{\pi/4}^{\pi/2} \cos\theta d\theta = \sin\theta\Big|_{\pi/4}^{\pi/2} = \sin\dfrac{\pi}{2} - \sin\dfrac{\pi}{4} = 1 - \dfrac{\sqrt{2}}{2}$

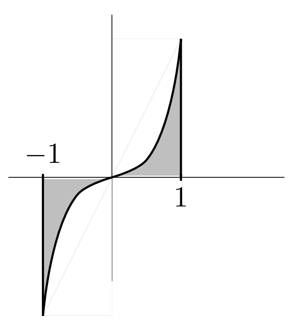
Return to Problem

13A.3. $\int \dfrac{dx}{e^{-x}} = \int e^x\,dx = e^x + c$

Return to Problem

13A.4. $\int_{-1}^{1} t^3 dt = \dfrac{t^4}{4}\Big|_{-1}^{1} = \dfrac{(1)^4}{4} - \dfrac{(-1)^4}{4} = \dfrac{1}{4} - \dfrac{1}{4} = 0$

Question: If a definite integral measures the area under the curve, why does it appear there is no area under t^3 over $[-1, 1]$?
(See Review Topic 13D.)



Return to Problem

Can you find

B.1. $\displaystyle\int -2\sec^2 x\,dx$

B.2. $\displaystyle\int_{\pi/4}^{\pi/2} \frac{4}{\sin^2 x}\,dx$

B.3. $\displaystyle\int e^{x+1}\,dx$

Answers:

B.1. $\displaystyle\int -2\sec^2 x\,dx = -2\int \sec^2 x\,dx = -2\tan x + c$

Return to Review Topic

B.2.
$$\begin{aligned}\int_{\pi/4}^{\pi/2} \frac{4}{\sin^2 x}\,dx &= 4\int_{\pi/4}^{\pi/2} \csc^2 x\,dx \\ &= 4(-\cot x)\Big|_{\pi/4}^{\pi/2} = 4\left[-\cot\frac{\pi}{2} - (-\cot\frac{\pi}{4})\right] \\ &= 4[-0+1] = 4\end{aligned}$$

Return to Review Topic

B.3. $\displaystyle\int e^{x+1}\,dx = \int e^x e\,dx = e\int e^x\,dx = e^{x+1} + c$

Return to Review Topic

13B.1. $$\int_{\pi/6}^{\pi/4} 4(\cos x \tan x + 1)\,dx = 4\int_{\pi/6}^{\pi/4} (\sin x + 1)\,dx$$
$$= 4[-\cos x + x]\Big|_{\pi/6}^{\pi/4} = 4\left[-\cos\frac{\pi}{4} + \frac{\pi}{4} - \left(-\cos\frac{\pi}{6} + \frac{\pi}{6}\right)\right]$$
$$= -2\sqrt{2} + 2\sqrt{3} + \frac{\pi}{3}$$

Return to Problem

13B.2. $$\int \frac{x^3 + x^2 + 1}{x}\,dx = \int \left(x^2 + x + \frac{1}{x}\right)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + \ln|x| + c$$

Return to Problem

13B.3. $$\int_{\pi/6}^{\pi/3} (\cot^2 x + 1)\,dx = \int_{\pi/6}^{\pi/3} \csc^2 x\,dx$$
$$-\cot x\Big|_{\pi/6}^{\pi/3} = -\cot\frac{\pi}{3} - \left(-\cot\frac{\pi}{6}\right)$$
$$= -\frac{1}{\sqrt{3}} + \sqrt{3}$$

Return to Problem

13B.4. $$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + c$$

Return to Problem

13B.5. $$\int (e^x - \cos x)\, dx = e^x - \sin x + c$$

Return to Problem

13B.6. $$\int (x^2 - 1)^2\, dx = \int (x^4 - 2x^2 + 1)\, dx = \frac{x^5}{5} - \frac{2x^3}{3} + x + c$$

Return to Problem

13B.7. $$\int \frac{e^{2x} - e^x}{e^x}\, dx = \int (e^x - 1)\, dx = e^x - x + c$$

Return to Problem

C.1. What substitution do you think you would use with

$$\int \frac{1}{2}(x^3 + 3x - 1)^{-1/2}(3x^2 + 3)\,dx?$$

C.2. $\int e^{\sin x} \cos x\,dx$

C.3. $\int e^{2x^{1/2}} \frac{dx}{x^{1/2}}$

C.4. $\int e^{3x^2+2x-1}(6x + 2)\,dx$

Answers:

C.1. $u = x^3 + 3x - 1$ and $du = (3x^2 + 3)\,dx$

$$\int \frac{1}{2}u^{-1/2}\,du = \frac{1}{2}\frac{u^{1/2}}{\frac{1}{2}} + c = \sqrt{x^3 + 3x - 1} + c$$

Return to Review Topic

C.2. $u = \sin x$, $du = \cos x\,dx$

$$\int e^u\,du = e^u + c = e^{\sin x} + c$$

Return to Review Topic

C.3. $u = 2x^{1/2}$, $du = \dfrac{dx}{x^{1/2}}$

$$\int e^u\,du = e^u + c = e^{2x^{1/2}} + c$$

Return to Review Topic

C.4. $u = 3x^2 + 2x - 1$, $du = (6x + 2)\,dx$

$$\int e^u\,du = e^u + c = e^{3x^2+2x-1} + c$$

Return to Review Topic

C.5. $\displaystyle\int \sin(x^2+1)x\,dx$

C.6. $\displaystyle\int \sec^2(e^x+x)(e^x+1)\,dx$

C.7. $\displaystyle\int \cos(x^3+3x-1)(x^2+1)\,dx$

Answers:

C.5. $u = x^2+1,\ du = 2x\,dx$

$$\int \sin u\frac{du}{2} = \frac{1}{2}\int \sin u\,du = \frac{1}{2}(-\cos u)+c = -\frac{1}{2}\cos(x^2+1)+c$$

Return to Review Topic

C.6. $u = e^x+x,\ du = (e^x+1)\,dx$

$$\int \sec^2 u\,du = \tan u + c = \tan(e^x+x)+c$$

Return to Review Topic

C.7. $u = x^3+3x-1,\ du = (3x^2+3)\,dx = 3(x^2+1)\,dx$

$$\int \cos u\frac{du}{3} = \frac{1}{3}\int \cos u\,du = \frac{1}{3}\sin u + c$$

$$= \frac{1}{3}\sin(x^3+3x-1)+c$$

Return to Review Topic

C.8. $\int x\sqrt{x+1}\,dx = \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + c$

Return to Review Topic

C.9.

	a =	F(x) =
$\int \sin 2x\,dx$	$a = \frac{1}{2}$	$-\frac{1}{2}\cos 2x + c$
$\int e^{3x} dx$	$a = 3$	$\frac{1}{3}e^{3x} + c$
$\int e^{-x/2}\,dx$	$a = -\frac{1}{2}$	$-2e^{-x/2} + c$
$\int \sec 3x \tan 3x\,dx$	$a = 3$	$\frac{1}{3}\sec 3x + c$
$\int \cos \pi x\,dx$	$a = \pi$	$\frac{1}{\pi}\sin \pi x + c$
$\int \frac{dx}{x+2}\,dx$	—	$\ln\lvert x+2\rvert + c$
$\int \frac{dx}{2-x}\,dx$	—	$-\ln\lvert 2-x\rvert + c$

Return to Review Topic

13C.1. $u = x - 1$, $du = dx$

$$\int \frac{du}{u^{1/2}} = \int u^{-1/2}\,du = 2u^{1/2} + c = 2(x-1)^{1/2} + c$$

Return to Problem

13C.2. $u = x - 1$, $du = dx$. Limits: $x = 2 \Rightarrow u = 1$; $x = 3 \Rightarrow u = 2$.

$$\begin{aligned}\int_1^2 \frac{u+1}{u^{1/2}}\,du &= \int_1^2 (u^{1/2} + u^{-1/2})du \\ &= \frac{2}{3}u^{3/2} + 2u^{1/2}\Big|_1^2 \\ &= \frac{2}{3}(2\sqrt{2}) + 2\sqrt{2} - \left(\frac{2}{3} + 2\right) \\ &= \frac{10}{3}\sqrt{2} - \frac{8}{3} \\ &= \frac{1}{3}(10\sqrt{2} - 8)\end{aligned}$$

Return to Problem

13C.3. $u = x^2 - 1$, $du = 2x\,dx$

$$\begin{aligned}\int u^{1/2}\frac{du}{2} = \frac{1}{2}\int u^{1/2}du &= \frac{1}{3}u^{3/2} + c \\ &= \frac{1}{3}(x^2-1)^{3/2} + c\end{aligned}$$

Return to Problem

13C.4. No substitution needed! Rewrite as

$$\int (x^{3/2} - x^{1/2})\,dx = \frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} + c$$

Return to Problem

13C.5. $u = \dfrac{2x}{3}$; $du = \dfrac{2}{3}\,dx$. Limits: $x = \dfrac{\pi}{2} \Rightarrow u = \dfrac{\pi}{3}$; $x = 0 \Rightarrow u = 0$;

$$\begin{aligned}\frac{3}{2}\int_0^{\pi/3} \cos u\,du = \frac{3}{2}\sin u\Big|_0^{\pi/3} &= \frac{3}{2}\left(\sin\frac{\pi}{3} - \sin 0\right)\\ &= \frac{3\sqrt{3}}{4}\end{aligned}$$

Return to Problem

13C.6. $u = e^x + 1$, $du = e^x\,dx$

$$\int \frac{du}{u} = \ln|u| + c = ln|e^x + 1| + c$$

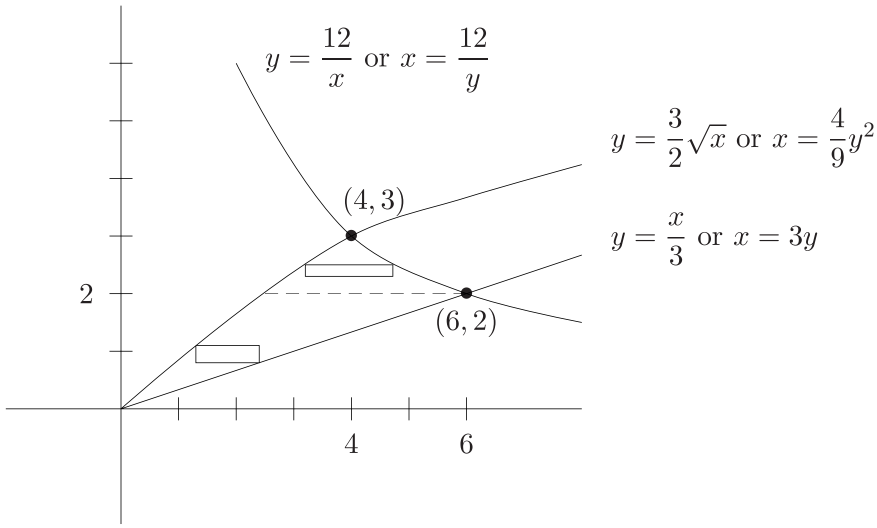
Return to Problem

13C.7. $u = 2x$, $du = 2\,dx$. Limits: $x = \dfrac{\pi}{6} \Rightarrow u = \dfrac{\pi}{3}$; $x = 0 \Rightarrow u = 0$.

$$\begin{aligned}\int_0^{\pi/3} \tan^2 u\frac{du}{2} &= \frac{1}{2}\int_0^{\pi/3} \tan^2 u\,du\\ &= \frac{1}{2}\int_0^{\pi/3} (\sec^2 u - 1)\,du = \frac{1}{2}(\tan u - u)\Big|_0^{\pi/3}\\ &= \frac{1}{2}\left[\left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \tan 0\right]\\ &= \frac{1}{2}\left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{1}{6}(3\sqrt{3} - \pi)\end{aligned}$$

Return to Problem

D.1. The area is described by Fig. 13D.10, but now we must use horizontal rectangles. Again, there are two possible descriptions. Also, as in Example 13D.4, we must solve for x in terms of y for all the curves.



The length of the upper rectangle is "right curve − left curve" $= \left(\frac{12}{y} - \frac{4}{9}y^2\right)$ for $2 \le y \le 3$, and the length of the lower rectangle is $\left(3y - \frac{4}{9}y^2\right)$ for $0 \le y \le 2$. Thus,

$$\begin{aligned}
\text{Area} &= \int_0^2 \left(3y - \frac{4}{9}y^2\right) dy + \int_2^3 \left(\frac{12}{y} - \frac{4}{9}y^2\right) dy \\
&= \left(\frac{3y^2}{2} - \frac{4y^3}{27}\right)\Bigg|_0^2 + \left(12\ln y - \frac{4y^3}{27}\right)\Bigg|_2^3 \\
&= \left(6 - \frac{32}{27}\right) - 0 + (12\ln 3 - 4) - \left(12\ln 2 - \frac{32}{27}\right) \\
&= 2 + 12\ln\left(\frac{3}{2}\right),
\end{aligned}$$

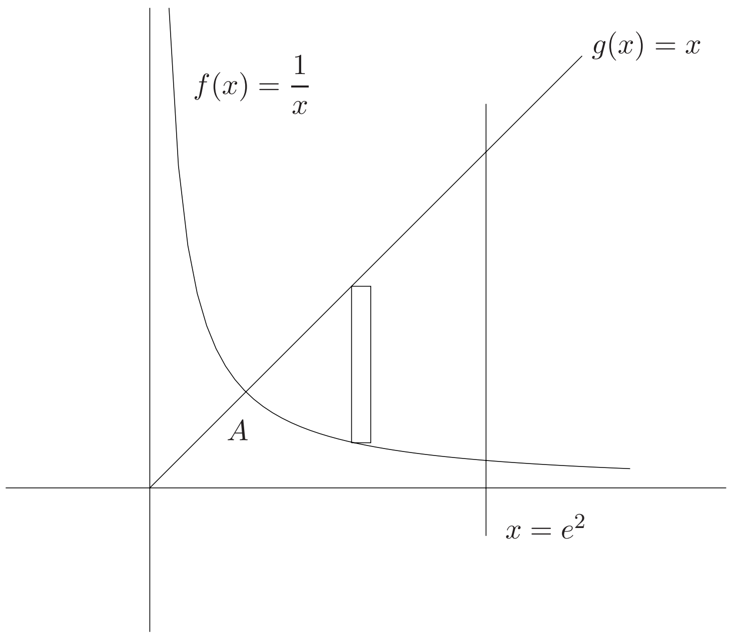
which is the same answer obtained in Example 13D.5 using vertical rectangles.

Return to Review Topic

13D.1. $\text{Area} = \int_{\pi/6}^{\pi/3} \cos x \, dx = \sin x \Big|_{\pi/6}^{\pi/3} = \sin \frac{\pi}{3} - \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}$

Return to Problem

13D.2.

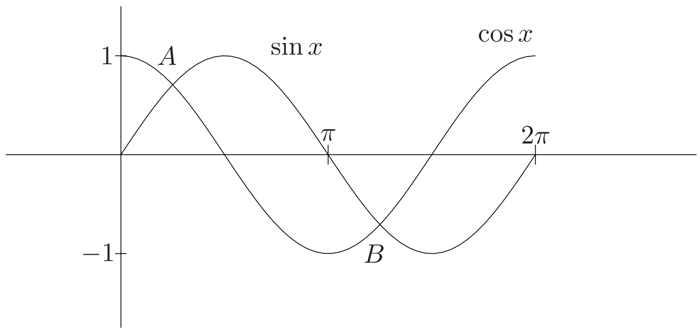


To find the point A, set $f(x) = g(x) \Rightarrow \frac{1}{x^2} = x \Rightarrow x^3 = 1 \Rightarrow x = 1$, $y = 1$.

$$\begin{aligned}
\text{Area} &= \text{“sum of rectangles”} \\
&= \int_1^{e^2} \left(x - \frac{1}{x^2} \right) dx = \frac{x^2}{2} + \frac{1}{x} \Big|_1^{e^2} \\
&= \frac{1}{2}(e^2)^2 + \frac{1}{e^2} - \left(\frac{1^2}{2} + \frac{1}{1} \right) \\
&= \frac{e^4}{2} + e^{-2} - \frac{3}{2}
\end{aligned}$$

Return to Problem

13D.3.

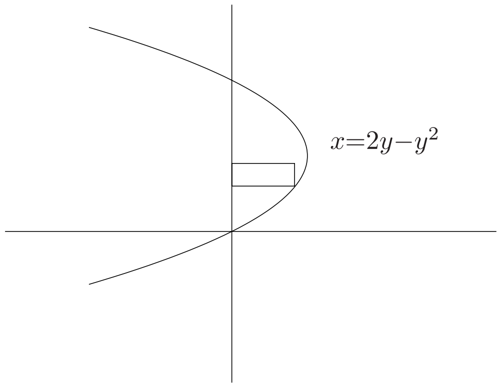


To find the x coordinates of the points A and B, set $\sin x = \cos x \Rightarrow x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. Since the area involves the "upper curve − lower curve", three integrals are necessary (see Example 13D.3). Thus,

$$
\begin{aligned}
\text{area} &= \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx \\
&\qquad + \int_{5\pi/4}^{2\pi} (\cos x - \sin x)\,dx \\
&= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)\Big|_{5\pi/4}^{2\pi} \\
&= \sin\frac{\pi}{4} + \cos\frac{\pi}{4} - (\cancelto{0}{\sin 0} + \cos 0) \\
&\qquad - \left[\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4} - \left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right)\right] \\
&\qquad + (\cancelto{0}{\sin 2\pi} + \cos 2\pi) - \left(\sin\frac{5\pi}{4} + \cos\frac{5\pi}{4}\right) \\
&= \sqrt{2} - 1 - [-\sqrt{2} - (\sqrt{2})] + 1 - (-\sqrt{2}) = 4\sqrt{2}
\end{aligned}
$$

Return to Problem

13D.4. In this problem, it is easier to use horizontal rectangles like Example 13D.4.



First, find where the curve intersects the y-axis.

$$\begin{aligned} 2y - y^2 &= 0 \\ y(2-y) &= 0 \\ y = 0,\ y &= 2 \end{aligned}$$

$$\begin{aligned} \text{Area} &= \int_0^2 [(2y - y^2) - 0]dy \\ &= y^2 - \frac{y^3}{3}\bigg|_0^2 \\ &= 4 - \frac{8}{3} - (0 - 0) \\ &= \frac{4}{3} \end{aligned}$$

Return to Problem

E.1-I.

$$\int \frac{x-1}{x^2}\,dx = \int \left(\frac{x}{x^2} - \frac{1}{x^2}\right) dx = \int \left(\frac{1}{x} - x^{-2}\right) dx = \ln|x| + x^{-1} + c$$

and

$$\int \frac{x^2}{x-1}\,dx = \int \left(x + 1 + \frac{1}{x-1}\right) dx = \frac{1}{2}x^2 + x + \ln|x-1| + c$$

for help with long division, see Review Topic 2.

Return to Problem

E.1-II.

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^{x^2} + c$$

$$u = x^2, \quad du = 2x\,dx$$

and

$$\int xe^{2x}\,dx$$

This integral can't be evaluated until you learn (in Calc II) a method called Integration by Parts.

Return to Problem

E.1-III.
$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{du}{u}\,du \qquad u = 1+e^x,\quad du = e^x\,dx$$
$$= \ln|1+e^x| + c$$
$$= \ln(1+e^x) + c$$

Why can the absolute value symbol be dropped?

and

$$\int \frac{2}{1+e^x}\,dx = \int \frac{2}{1+e^x}\cdot\frac{e^{-x}}{e^{-x}}\,dx = \int \frac{2e^{-x}}{e^{-x}+1}\,dx$$
$$= -2\int \frac{du}{u} \qquad u = e^{-x}+1$$
$$= -2\ln|e^{-x}+1| + c$$
$$= -2\ln(e^{-x}+1) + c$$

For more examples on multiplying by a form of 1, see Review Topic 2.

Return to Problem

E.1-IV.
$$\int \frac{\cos x}{1+\sin x}\,dx = \int \frac{du}{u} \qquad u = 1+\sin x,\quad du = \cos x\,dx$$
$$= \ln|1+\sin x| + c$$
$$= \ln(1+\sin x) + c$$

and

$$\int \frac{\cos^2 x}{1+\sin x}\,dx = \int \frac{1-\sin^2 x}{1+\sin x}\,dx = \int (1-\sin x)\,dx$$
$$= x + \cos x + c$$

Return to Problem