MATH 250 – REVIEW TOPIC 7

Polar Coordinates

Polar coordinates represent another way to locate a point in the coordinate plane. The method with which we are most familiar is the Cartesian or rectangular system. This system utilizes a horizontal or x-axis and a vertical or y-axis. For example, to plot A in rectangular coordinates in Fig. 7.1, we start from the origin $(0,0)$ and "go over to the right 2 and up 2". Thus $A=(2,2)$ in rectangular coordinates.

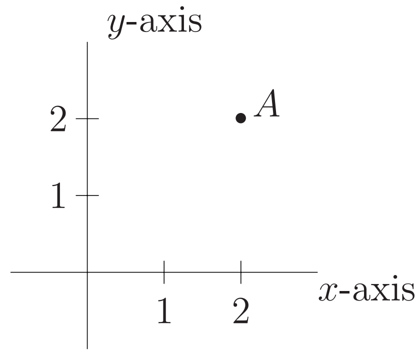

Fig. 7.1

Is there another way to locate A? Yes! We could start from the horizontal axis and rotate a ray in the counterclockwise direction a certain angle θ. (Rotating clockwise gives a negative value for θ.) We then travel out the ray a positive distance r until we hit A. So, rotating through an angle θ and traveling a distance r will also plot the point A. The pair (r,θ) represents the polar coordinates of A. Note that in polar coordinates, the origin is called the pole and the x-axis is called the polar axis. How can the rectangular coordinates of the point A be changed into polar coordinates? Looking at Fig. 7.2 and using right-angle trigonometry (Math 150, Review Topic 9), we have the following formulas denoted by Eqns 7.1.

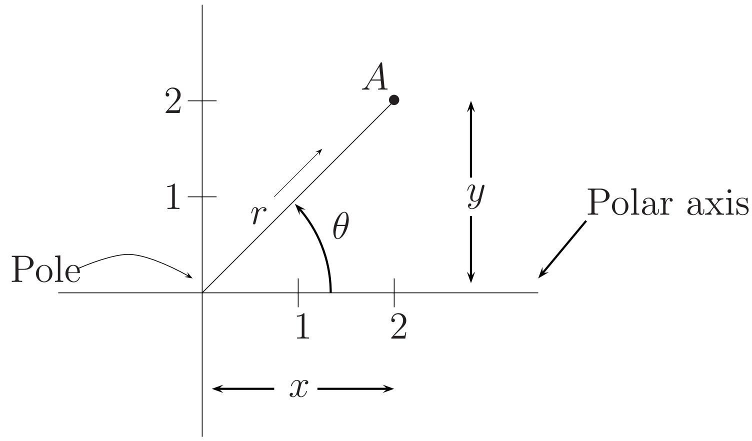

Fig. 7.2

$$\begin{aligned} &\cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r}, \quad \tan\theta = \frac{y}{x} \\ &x^2 + y^2 = r^2 \quad \text{Pythagorean Theorem} \end{aligned} \tag{7.1}$$

Since $x = y = 2$, $r = 2\sqrt{2}$ and $\theta = \dfrac{\pi}{4}$. In polar coordinates then, A can be written as $\left(2\sqrt{2}, \dfrac{\pi}{4}\right)$.

A moment's thought reveals that there are other ways to write A in polar coordinates. For example, A is also given by (i) $\left(2\sqrt{2}, 2\pi + \dfrac{\pi}{4}\right)$, (ii) $\left(2\sqrt{2}, \dfrac{-7\pi}{4}\right)$, or (iii) $\left(-2\sqrt{2}, \dfrac{5\pi}{4}\right)$. In (i), any multiple of 2π would also work $\left(\theta = 4\pi + \dfrac{\pi}{4}, -6\pi + \dfrac{\pi}{4}, \text{ etc.}\right)$. In (ii), notice that $\theta = \dfrac{-7\pi}{4}$ is

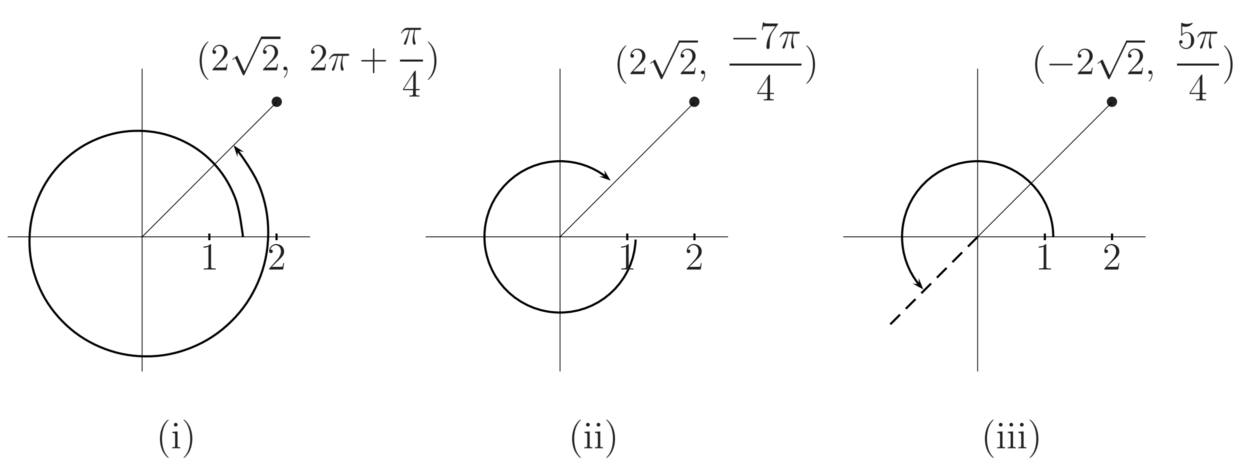

Fig. 7.3

negative. This means θ starts from the polar axis and swings in the clockwise direction. In (iii), r is negative. What does this mean? A negative value of r means that after you swing through the value θ, you go out from the origin or pole in the opposite direction. For example, $\left(-2\sqrt{2}, \dfrac{-3\pi}{4}\right)$ is another way to write A in polar coordinates.

As you can see there are an infinite number of ways to represent the same point in polar coordinates. This is not the case for rectangular coordinates. For example, the point $(3, 2)$ can be written in only one way in rectangular coordinates.

Polar coordinates are very important when studying physical problems and often lead to simpler equations. The circle $x^2 + y^2 = 4$ can be described in polar coordinates by the simple equation $r = 2$. (See Example 8.3 in Review Topic 8.)

Exercise 7.1: (i) Plot the following points in polar coordinates.

(a) $\left(2, \dfrac{\pi}{6}\right)$; (b) $\left(2, \dfrac{7\pi}{6}\right)$; (c) $\left(2, \dfrac{5\pi}{6}\right)$; (d) $\left(2, \dfrac{-5\pi}{6}\right)$;
(e) $\left(2, \dfrac{-7\pi}{6}\right)$; (f) $\left(-2, \dfrac{\pi}{6}\right)$; (g) $\left(-2, \dfrac{-5\pi}{6}\right)$. Answers

(ii) Which of the above pairs represent the same point?

The formulas from (7.1) actually hold for any quadrant. They lead to the following.

$$x = r\cos\theta, \quad y = r\sin\theta \tag{7.2}$$

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x} \tag{7.3}$$

Given (r, θ), you can use (7.2) to get (x, y). Similarly, given (x, y), use (7.3) to get (r, θ). In Calc II, it is sometimes useful to go back and forth between the two systems.

Example 7.1. Change the following points from rectangular to polar coordinates. It is customary to write polar coordinates with $r > 0$ and $0 \le \theta < 2\pi$.

$$(1, \sqrt{3}), \quad (-1, 0), (-\sqrt{3}, -1).$$

Using (3), $(1, \sqrt{3}) \Rightarrow r^2 = 1^2 + (\sqrt{3})^2 = 4 \Rightarrow r = 2.$

$$\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}$$
$$\Rightarrow (1, \sqrt{3})_R = \left(2, \frac{\pi}{3}\right)_P.$$

By inspection, $(-1, 0)_R = (1, \pi)_P$. (Verification of this also follows from (3).)

Aside: $(-1, 0)_R = (-1, 0)_P$ as well. (You may have to think about this!)

Next, $(-\sqrt{3}, -1)_R \Rightarrow r^2 = (-\sqrt{3})^2 + (-1)^2 = 4 \Rightarrow r = 2.$

$$\tan\theta = \frac{-1}{-\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \, ?$$

Since our point is in the third quadrant, we choose $\theta = \frac{7\pi}{6}$ (not $\frac{\pi}{6}$).

$$\Rightarrow (-\sqrt{3}, -1)_R = \left(2, \frac{7\pi}{6}\right)_P.$$

Example 7.2. Change the following points from polar to rectangular coordinates $\left(2, \frac{\pi}{6}\right)$, $\left(-1, \frac{\pi}{2}\right)$.

Using (7.2),

$$x = 2\cos\frac{\pi}{6} = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$
$$y = 2\sin\frac{\pi}{6} = 2 \cdot \frac{1}{2} = 1$$
$$\Rightarrow \left(2, \frac{\pi}{6}\right)_P = (\sqrt{3}, 1)_R.$$

By inspection, $\left(-1, \frac{\pi}{2}\right)_P = (0, -1)_R$. (Verify that this also follows from (7.2).)

Exercise 7.2: Change the following points from rectangular to polar coordinates.

(i) $(-2, 2)$ (ii) $(3\sqrt{3}, 3)$ (iii) $(1, -1)$ Answers

Exercise 7.3: Change the following points from polar to rectangular coordinates.

(i) $\left(\dfrac{1}{2}, \dfrac{\pi}{6}\right)$ (ii) $\left(2, \dfrac{4\pi}{3}\right)$ Answers

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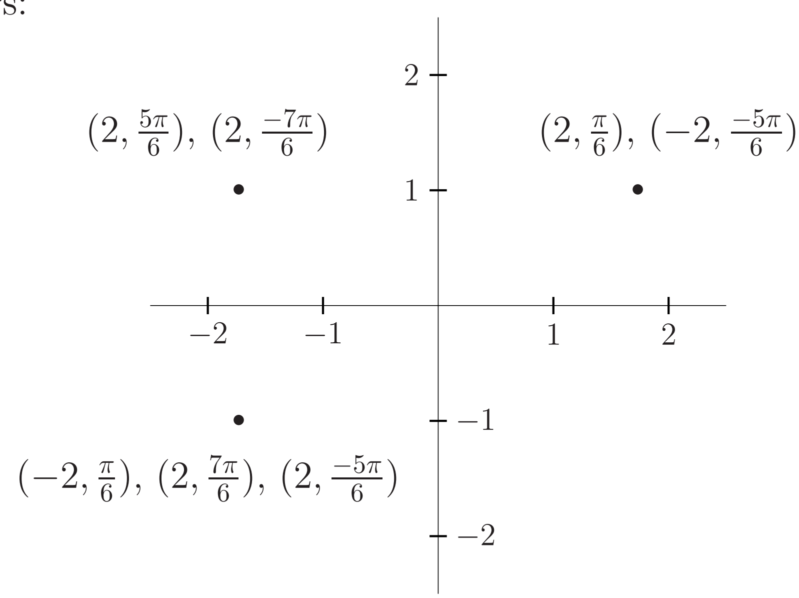
(i) Plot the following points in polar coordinates.

(a) $\left(2, \frac{\pi}{6}\right)$; (b) $\left(2, \frac{7\pi}{6}\right)$; (c) $\left(2, \frac{5\pi}{6}\right)$; (d) $\left(2, \frac{-5\pi}{6}\right)$;
(e) $\left(2, \frac{-7\pi}{6}\right)$; (f) $\left(-2, \frac{\pi}{6}\right)$; (g) $\left(-2, \frac{-5\pi}{6}\right)$.

Answers:



(ii) Which of the above pairs represent the same point?

Answers:

(a) and (g) are the same point;
(b), (d), and (f) are the same point;
(c) and (e) are the same point.

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Change the following points from rectangular to polar coordinates.

(i) $(-2, 2)$ (ii) $(3\sqrt{3}, 3)$ (iii) $(1, -1)$

Answers:

(i) $(-2, 2)_R \Rightarrow x = -2,\ y = 2 \Rightarrow r = \sqrt{(-2)^2 + (2)^2} = 2\sqrt{2}$.

$\tan\theta = \dfrac{y}{x} = \dfrac{2}{-2} = -1 \Rightarrow \theta = \dfrac{3\pi}{4}$, since $(-2, 2)$ is in the second quadrant.

$$\Rightarrow (-2, 2)_R = \left(2\sqrt{2}, \frac{3\pi}{4}\right)_P.$$

(ii) $(3\sqrt{3}, 3)_R \Rightarrow r = \sqrt{(3\sqrt{3})^2 + 3^2} = 6$.

$\tan\theta = \dfrac{3}{3\sqrt{3}} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \dfrac{\pi}{6}$, since $(3\sqrt{3}, 3)$ is in the first quadrant.

$$\Rightarrow (3\sqrt{3}, 3)_R = \left(6, \frac{\pi}{6}\right)_P.$$

(iii) $(1, -1)_R \Rightarrow r = \sqrt{1^2 + (-1)^2} = \sqrt{2}$.

$\tan\theta = \dfrac{-1}{1} = -1 \Rightarrow \theta = \dfrac{7\pi}{4}$, since $(1, -1)$ is in the fourth quadrant.

$$\Rightarrow (1, -1)_R = \left(1, \frac{7\pi}{4}\right)_P \quad \left(\text{also} = \left(1, \frac{-\pi}{4}\right)_P\right).$$

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Change the following points from polar to rectangular coordinates.

(i) $\left(\frac{1}{2}, \frac{\pi}{6}\right)$ (ii) $\left(2, \frac{4\pi}{3}\right)$

Answers:

(i) $$\left(\frac{1}{2}, \frac{\pi}{6}\right)_P \Rightarrow x = \frac{1}{2}\cos\frac{\pi}{6} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4}$$
$$y = \frac{1}{2}\sin\frac{\pi}{6} = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$
$$\Rightarrow \left(\frac{1}{2}, \frac{\pi}{6}\right)_P = \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right)_R .$$

(ii) $$\left(2, \frac{4\pi}{3}\right)_P \Rightarrow x = 2\cos\frac{4\pi}{3} = 2\cdot\left(\frac{-1}{2}\right) = -1$$
$$y = 2\sin\frac{4\pi}{3} = 2\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{3}$$
$$\Rightarrow \left(2, \frac{4\pi}{3}\right)_P = (-1, -\sqrt{3})_R .$$

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