MATH 250 – REVIEW TOPIC 9

Trigonometric Substitutions

In Calc I you were introduced to the concept of using substitutions to evaluate integrals. This section discusses certain trig substitutions that will be extremely useful when studying integration techniques in Calc II. The following examples illustrate the main idea: use trig identities to make two terms reduce to one.

Consider the expression $\sqrt{1-x^2}$. Is there some way to manipulate this expression so that the radical disappears and something simple is left? One way to make the radical disappear is to collapse the two terms under the radical sign to one term (this is the key!). This will happen if we let $x = \sin\theta$. This is called a trig substitution for x. Then $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$ (the two terms collapse to one). Thus,

$$\left.\begin{array}{l}\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta \\ \text{Trig substitution: } x = \sin\theta,\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}\end{array}\right\} \tag{9.1}$$

Two Questions: Why did we say $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$, and why is $\sqrt{\cos^2\theta} = \cos\theta$ and not $|\cos\theta|$?

First, $\sqrt{1-x^2}$ requires $x^2 \le 1$ or $-1 \le x \le 1$. If $x = \sin\theta$, then $-1 \le x \le 1$ matches $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$. For these values of θ, $\cos\theta \ge 0$. Thus, $|\cos\theta| = \cos\theta$.

Note: In (9.1) we could have made the substitution $x = \cos\theta$, $0 \le \theta \le \pi$, but opted for setting $x = \sin\theta$ instead.

Next, consider $\sqrt{1+x^2}$. What trig substitution for x will lead to a simpler expression? If we let $x = \tan\theta$, then $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$ (again the two terms collapse to one). Thus

$$\left.\begin{array}{l}\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta \\ \text{Trig substitution: } x = \tan\theta,\ -\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}\end{array}\right\} \tag{9.2}$$

For (9.1), $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$. For (9.2), we wrote instead $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. Why?

The expression $\sqrt{1+x^2}$ allows x to assume any value. By looking at the graph of $x = \tan\theta$ (Math 150, Review Topic 13), we see that x assuming any value is equivalent to θ being in the interval $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$. For θ in this interval $\sec\theta > 0$, and so $\sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$.

As a third example, consider $\sqrt{x^2-1}$. What trig substitution for x will make this collapse to one term so that we can "get rid of" the square root? As you probably guessed, let $x = \sec\theta$. Then

$$\left.\begin{array}{c} \sqrt{x^2-1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta \\ \text{Trig substitution: } x = \sec\theta,\ 0 \le \theta < \dfrac{\pi}{2} \ \text{ or } \ \pi \le \theta < \dfrac{3\pi}{2} \end{array}\right\} \tag{9.3}$$

In order to understand the intervals for θ as stated in (9.3), we need to review the graphs of $\sec\theta$ and $\tan\theta$. The expression $\sqrt{x^2-1}$ requires $x^2 \ge 1 \Rightarrow |x| \ge 1 \Rightarrow x \ge 1$ or $x \le -1$. For $x \ge 1$, $\sec\theta$ has the domain $0 \le \theta < \dfrac{\pi}{2}$. For $x \le -1$, $\sec\theta$ has the domain $\pi \le \theta < \dfrac{3\pi}{2}$. On these θ intervals, $\tan\theta \ge 0$. Thus $\sqrt{\tan^2\theta} = |\tan\theta| = \tan\theta$.

Equations (9.1), (9.2), and (9.3) illustrate the basic trig substitutions. In each case, setting $x =$ a trigonometric function of θ in a certain interval causes two terms to collapse to one and leads to a simpler representation. Equation (9.1) is based on the identity $1 - \sin^2\theta = \cos^2\theta$, (9.2) on the identity $1 + \tan^2\theta = \sec^2\theta$, and (9.3) on $\sec^2\theta - 1 = \tan^2\theta$.

Notice that the starting expression containing x in (9.2) has a "+" sign between 1 and x^2, while (9.1) and (9.3) have minus signs between 1 and x^2. Also, for (9.1) the x term is <u>after</u> the minus sign whereas for (9.3), the x term is in <u>front</u> of the minus sign.

How can these substitutions be extended to more complicated situations? We will give a partial answer to this question. The complete answer will come in Calc II.

Example 9.1. Consider the expression $(4-x^2)^{3/2}$. What trig substitution for x works here? Should we try the form (9.1), (9.2), or (9.3)?

Since the variable is <u>after</u> the minus sign, our problem fits the form (9.1). However, $x = \sin\theta$ won't work. We need a coefficient in front of $\sin\theta$. So, we set $x = (?)\sin\theta$ and find the coefficient (?) as follows.

$$4 - x^2 = 4\left(1 - \frac{x^2}{4}\right) = 4\left(1 - \left(\frac{x}{2}\right)^2\right) = 4(1 - (\quad)^2).$$

Notice that we manipulated to get the "1" in $(1-(\quad)^2)$. Next we let $(\quad) = \dfrac{x}{2} = \sin\theta$ which implies

$$x = 2\sin\theta. \tag{9.4}$$

Then $(4-x^2)^{3/2} = (4 - 4\sin^2\theta)^{3/2} = \left(4(1-\sin^2\theta)\right)^{3/2} = (4\cos^2\theta)^{3/2} = 8\cos^3\theta$. (The algebra involving substitutions like this is introduced in Review Topic 4.)

Exercise 9.1: Simplify the following expressions using trig substitutions of the form (9.1), (9.2), or (9.3).

(a) $(x^2 + 9)$ (b) $(2 - x^2)^{3/2}$ Answers

(c) $(x^2 - 5)^{3/2}$ (d) $\sqrt{1 - 4x^2}$ (e) $\sqrt{16 - 81x^2}$ Answers

Hint: for (e) let $x = \dfrac{4}{9}\sin\theta$. How would you determine x without the hint?

Let's return to Example 9.1. We began with $\sqrt{4 - x^2}$ and made the trig substitution $x = 2\sin\theta$. In the course of evaluating an integral in Calc II using this trig substitution, you might have to find an expression for $\tan\theta$ or $\cos\theta$ in terms of x, where $x > 0$. How would you do this? Easy! Since $x > 0$, $x = 2\sin\theta$ means $\sin\theta = \dfrac{x}{2} = \dfrac{\text{opp.}}{\text{hyp.}}$. This gives the following picture.

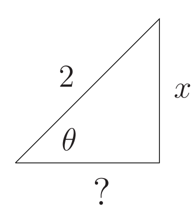


Fig. 9.1.

By the Pythagorean Theorem, the unknown third side is $\sqrt{4 - x^2}$. Thus,
$\tan\theta = \dfrac{x}{\sqrt{4-x^2}}$, $\cos\theta = \dfrac{\sqrt{4-x^2}}{2}$, etc.

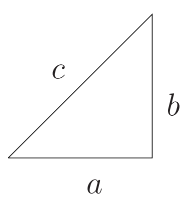
Exercise 9.2: Exercise 9.1(i) asks you to simplify the expression $x^2 + 9$. This requires the trig substitution $x = 3\tan\theta$. Given this substitution, find $\sin\theta$ and $\cos\theta$ in terms of x. Assume $x > 0$. Answer

Exercise 9.3: (Optional!) The following integral is a type you will study in Calc II. (We thought you might be interested in seeing how a trigonometric substitution actually works in integration.) If you have some time, try it. Once you determine the trig substitution, the rest of the problem follows from Calc I concepts.

Evaluate $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$. Answer

Alternate Method for Trig Substitutions

There is an alternate method for determining what trig substitution of the form (9.1), (9.2), or (9.3) is correct when we know $x > 0$. It is based on Fig. 9.2 below.



; Pythag. Thm. $c^2 = a^2 + b^2$

$\rightarrow$ $c = \sqrt{a^2 + b^2}$

$\rightarrow$ $b = \sqrt{c^2 - a^2}$

Since a and b are interchangeable, we do not need a third form.

Fig. 9.2.

Consider the expression $\sqrt{9 - x^2}$. Since the variable is after the minus sign, the correct substitution has the form (9.1); i.e., $x = (?)\sin\theta$. What is the coefficient? Well, $(9 - x^2)$ is the difference of two squares (like $c^2 - b^2$ in Fig. 9.2). Think of 9 as the square of the hypotenuse and the variable term x^2 as the square of the vertical leg. So we have the picture below.

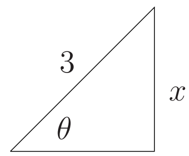
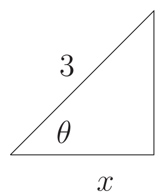


Fig. 9.3.

Since we are interested in $\sin\theta$, we write $\sin\theta = \dfrac{x}{3}$. Solving for x yields $x = 3\sin\theta$, which is the desired trig substitution. Then

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9\cos^2\theta} = 3\cos\theta.$$

Remark 1: In the expression $\sqrt{9-x^2}$, we said to think of 9 as the square of the hypotenuse and x^2 as the square of the vertical leg. What if we tried to make x^2 match the horizontal leg, as below?



This would lead to $\cos\theta = \dfrac{x}{3}$ or $x = 3\cos\theta$, which would work. However, we said earlier that we opted to use a substitution based on $\sin\theta$. To get a substitution involving $\sin\theta$, x must match the <u>vertical</u> leg as in Fig. 9.3.

Remark 2: The third side of the triangle in Fig. 9.3 is $\sqrt{9-x^2}$, which is our starting expression. This is no accident! It also means that Fig. 9.1 and Fig. 9.3 describe the same type of situation.

Example 9.3. What is the correct trig substitution for x to simplify $(x^2-7)^{3/2}$?

This example has the form (9.3), a variable followed by a minus sign. (This means our substitution has the form $x = (?)\sec\theta$.) As above, we match x^2 and 7 with the sides of a right triangle. Because of the minus sign, x^2 should represent the square of the hypotenuse, but this time 7 is the square of the <u>horizontal</u> leg (like $c^2 - a^2$ in Fig. 9.2). Since we are interested in $\sec\theta$, we write

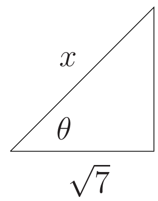


Fig. 9.4.

Then $\sec\theta = \dfrac{\text{hyp.}}{\text{adj.}} = \dfrac{x}{\sqrt{7}}$, or

$$x = \sqrt{7}\sec\theta \quad \text{(The desired trig substitution).}$$

So,

$$\begin{aligned}(x^2-7)^{3/2} &= \left(\left(\sqrt{7}\sec\theta\right)^2 - 7\right)^{3/2} = (7\sec^2\theta - 7)^{3/2} \\ &= (7\tan^2\theta)^{3/2} = 7^{3/2}\tan^3\theta.\end{aligned}$$

Remark 3: Letting 7 be the square of the <u>vertical</u> leg in Fig. 9.5 leads to a substitution involving $\csc\theta$.

We present one more example.

Example 9.4. Simplify $(1+2x^2)^3$ with a trig substitution.

This problem fits the form (9.2) since there is a plus sign between 1 and $2x^2$. Once again, we match 1 and $2x^2$ with the sides of a right triangle. The plus sign in $(1+2x^2)$ means that 1 and $2x^2$ correspond to the legs. Since we are interested in a substitution of the form (9.2) which involves $\tan\theta$ and since $\tan\theta = \text{opp/adj}$, we let $2x^2$ represent the square of the <u>vertical</u> (opp) leg and 1 represent the square of the <u>horizontal</u> (adj) leg as in Fig. 9.5. (Reversing this would lead to a substitution involving $\cot\theta$.) Our picture now becomes

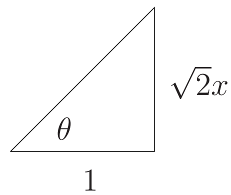


Fig. 9.5.

Fig. 9.5 implies

$$\tan\theta = \sqrt{2}x, \quad \text{or} \quad x = \frac{\tan\theta}{\sqrt{2}}.$$

This is our desired trig substitution which should simplify our starting expression. Let's plug it in and see.

$$\begin{aligned}(1+2x^2)^3 &= \left(1 + 2\left(\frac{\tan\theta}{\sqrt{2}}\right)^2\right)^3 \\ &= (1+\tan^2\theta)^3 = (\sec^2\theta)^3 = \sec^6\theta.\end{aligned}$$

As in all the previous examples, our trig substitution has reduced two terms raised to a power to one term raised to a power.

Exercise 9.4: Simplify the following expressions using the Alternate Method and trig substitutions of the form (9.1), (9.2), or (9.3).

a) $\sqrt{5-x^2}$ (b) $(3x^2+4)^{3/2}$ Answers

Simplify the following expressions using trig substitutions of the form (9.1), (9.2), or (9.3).

(a) $(x^2 + 9)$ (b) $(2 - x^2)^{3/2}$

Answers:

a) This corresponds to the form (9.2). We let $x = (\ \ )\tan\theta$ and solve for the coefficient ( ) by first manipulating to isolate the "1", since we want to match the form $\tan^2\theta + 1$. Thus,

$$x^2 + 9 = 9\left(\frac{x^2}{9} + 1\right) = 9(\tan^2\theta + 1)$$

$$\Rightarrow \tan^2\theta = \frac{x^2}{9} \Rightarrow \tan\theta = \frac{x}{3} \Rightarrow x = 3\tan\theta.$$

Then,

$$x^2 + 9 = (3\tan\theta)^2 + 9 = 9\tan^2\theta + 9 = 9(\tan^2\theta + 1) = 9\sec^2\theta.$$

b) This has the form (9.1). Again, we want to isolate the "1" first. Thus,

$$2\left(1 - \frac{x^2}{2}\right) = 2(1 - \sin^2\theta) \Rightarrow \sin^2\theta = \frac{x^2}{2} \Rightarrow x = \sqrt{2}\sin\theta.$$

This yields:

$$(2 - x^2)^{3/2} = \left(2 - (\sqrt{2}\sin\theta)^2\right)^{3/2} = (2 - 2\sin^2\theta)^{3/2}$$

$$= (2\cos^2\theta)^{3/2} = 2^{3/2}\cos^3\theta.$$

(Answers for c), d), and e) are on the next page –>)

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Simplify the following expressions using trig substitutions of the form (9.1), (9.2), or (9.3).

(c) $(x^2-5)^{3/2}$ (d) $\sqrt{1-4x^2}$ (e) $\sqrt{16-81x^2}$

Hint: for (e) let $x = \dfrac{2}{\sqrt{3}}\sin\theta$. How would you determine x without the hint?

Answers:

c) This has the form (9.3). So we write

$$x^2-5 = 5\left(\frac{x^2}{5}-1\right) = 5(\sec^2\theta - 1) \Rightarrow \frac{x^2}{5} = \sec^2\theta \Rightarrow x = \sqrt{5}\sec\theta.$$

Then, if $x = \sqrt{5}\sec\theta$, we have

$$(x^2-5)^{3/2} = (5\sec^2\theta - 5)^{3/2} = (5\tan^2\theta)^{3/2} = 5^{3/2}\tan^3\theta.$$

d) This has the form (9.1), and so we want to arrive at something involving $1-\sin^2\theta$. To do this we need $4x^2 = \sin^2\theta$, or $x = \dfrac{\sin\theta}{2}$. Plugging this in we have:

$$\sqrt{1-4x^2} = \sqrt{1-4\left(\frac{\sin\theta}{2}\right)^2} = \sqrt{1-4\frac{\sin^2\theta}{4}} = \sqrt{\cos^2\theta} = \cos\theta.$$

e) The hint says let $x = \dfrac{4}{9}\sin\theta$. We get:

$$\sqrt{16-81x^2} = \sqrt{16-81\left(\frac{4}{9}\sin\theta\right)^2} = \sqrt{16-81\cdot\frac{16\sin^2\theta}{81}}$$
$$= \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\cos\theta.$$

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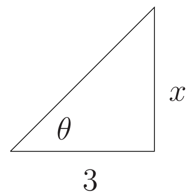
Exercise 9.1(i) asks you to simplify the expression $x^2 + 9$. This requires the trig substitution $x = 3\tan\theta$. Given this substitution, find $\sin\theta$ and $\cos\theta$ in terms of x. Assume $x > 0$.

Answer:

The subsitution $x = 3\tan\theta \Rightarrow \tan\theta = \dfrac{x}{3}$, or



This means the hypotenuse $= \sqrt{x^2 + 3}$. Thus

$$\sin\theta = \frac{x}{\sqrt{x^2+3}}, \quad \text{and} \quad \cos\theta = \frac{3}{\sqrt{x^2+3}}.$$

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(Optional!) Evaluate $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$.

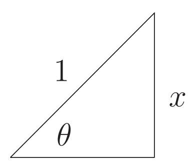
Answer:

(This is how you will use trig substitutions in Calc II.)

Let $x = \sin\theta$. Then $dx = \cos\theta d\theta$. Our integral becomes:

$$\begin{aligned}\int \frac{\sqrt{1-\sin^2\theta}\cos\theta d\theta}{\sin^2\theta} &= \int \frac{\cos^2\theta}{\sin^2\theta} d\theta = \int \cot^2\theta d\theta \\ &= \int (\csc^2\theta - 1) d\theta = -\cot\theta - \theta + c.\end{aligned}$$

Now we must put everything in terms of x. The substitution $x = \sin\theta$ has the following picture.



The missing side has length $\sqrt{1-x^2}$, which means $\cot\theta = \dfrac{\sqrt{1-x^2}}{x}$ and $\theta = \sin^{-1} x$. Thus

$$\int \frac{\sqrt{1-x^2}}{x^2}dx = -\cot\theta - \theta + c = -\frac{\sqrt{1-x^2}}{x} - \sin^{-1} x + c.$$

(See what you can look forward to in Calc II?!)

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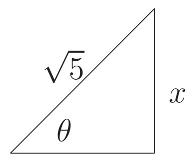
Simplify the following expressions using the Alternate Method and trig substitutions of the form (9.1), (9.2), or (9.3).

a) $\sqrt{5-x^2}$ (b) $(3x^2+4)^{3/2}$

Answers:

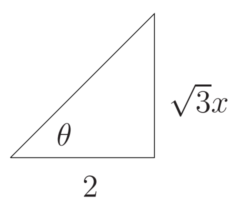
a) Simplify $\sqrt{5-x^2}$ using the Alternate Method. We draw a right triangle with $\sqrt{5}$ as the hypotenuse and x as the <u>vertical</u> leg.



We know our substitution has the form (9.1) which involves $\sin\theta$. According to the above picture, $\sin\theta = \dfrac{x}{\sqrt{5}}$ or $x = \sqrt{5}\sin\theta$. Using this as our substitution yields:

$$\sqrt{5-x^2} = \sqrt{5-(\sqrt{5}\sin\theta)^2} = \sqrt{5-5\sin^2\theta} = \sqrt{5\cos^2\theta} = \sqrt{5}\cos\theta.$$

b) Simplify $(3x^2+4)^{3/2}$ using the Alternate Method. We draw a right triangle with $\sqrt{3}x$ as the <u>vertical</u> leg and 2 as the horizontal leg.



Thus $\tan\theta = \dfrac{\sqrt{3}x}{2} \Rightarrow x = \dfrac{2\tan\theta}{\sqrt{3}}$. Plugging this in we have:

$$(3x^2+4)^{3/2} = \left(3\left(\frac{2\tan\theta}{\sqrt{3}}\right)^2 + 4\right)^{3/2} = \left(3\cdot\frac{4\tan^2\theta}{3} + 4\right)^{3/2}$$

$$= (4\tan^2\theta + 4)^{3/2} = (4\sec^2\theta)^{3/2} = 8\sec^3\theta.$$

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