Some New Results on Lyapunov-Type Diagonal Stability

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Consider the first-order linear constant coefficient system of \( n \) ordinary differential equations:

\[
\frac{dx}{dt} = A[x(t) - \hat{x}] \quad (1)
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( x(t), \hat{x} \in \mathbb{R}^n \).
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$\hat{x}$ is called an equilibrium for this system. If $x(t)$ converges to $\hat{x}$ as $t \to \infty$ for every choice of the initial data $x(0)$, the equilibrium $\hat{x}$ is said to be asymptotically stable.
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$\hat{x}$ is called an equilibrium for this system. If $x(t)$ converges to $\hat{x}$ as $t \to \infty$ for every choice of the initial data $x(0)$, the equilibrium $\hat{x}$ is said to be asymptotically stable.

The equilibrium is asymptotically stable if and only if each eigenvalue of $A$ has a negative real part. A matrix $A$ satisfying this condition is called a (Hurwitz) stable matrix.
Definition

Let $A \in \mathbb{R}^{n \times n}$ be symmetric matrix. Then $A$ is said to be positive semidefinite (positive definite) if $x^*Ax \geq 0$ ($x^*Ax > 0$) for all nonzero $x \in \mathbb{R}^n$.

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (positive definite) if and only if all of its eigenvalues are nonnegative (positive).

- A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite (positive definite) if and only if all its principal minors are nonnegative (positive).
  - The determinant of a principal submatrix is called a principal minor.

- We shall denote $A \succeq 0$ ($A \succ 0$) when $A$ is positive semidefinite (positive definite).
Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be (positive) stable if each eigenvalue of $A$ has a positive real part.
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Lyapunov’s Theorem

A matrix $A \in \mathbb{R}^{n \times n}$ is stable if and only if there exists a $P \succ 0$ such that

$$PA + A^T P \succ 0. \quad (2)$$

Then, $P$ is called a Lyapunov solution of (2).
Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be (Lyapunov) diagonally stable if there exists a positive diagonal matrix $D$ such that

$$DA + A^T D \succ 0.$$  \hfill (3)

Then, $D$ is called a diagonal (Lyapunov) solution of (3).
Definition

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Then, $D$ is called a diagonal (Lyapunov) solution of (3).

Definition

Let $A^{(1)}, A^{(2)}, \ldots, A^{(m)} \in \mathbb{R}^{n \times n}$. If there exists a positive diagonal matrix $D$ such that

$$DA^{(k)} + (A^{(k)})^T D \succ 0, \quad k = 1, 2, \ldots, m,$$  \hspace{1cm} (4)

then $D$ is called a common diagonal (Lyapunov) solution of (4). The existence of such a $D$ is interpreted as the simultaneous diagonal stability of $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$. 

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Lyapunov-Type Diagonal Stability

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Let \( A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \). The matrix \( A \) is stable, having the eigenvalues \( 1 \pm i \).

Choosing positive diagonal matrix \( D = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \), we have

\[
DA + A^T D = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \succ 0,
\]

thus \( A \) is a diagonally stable matrix.
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= \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix} \succ 0,
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thus \( A \) is a diagonally stable matrix.

Let \( B = \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} \). The matrix \( B \) is stable, having the eigenvalues \( 1 \pm i \).

However, \( B \) is not a diagonally stable matrix.

\[
DB + B^T D = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}
= \begin{bmatrix} 4d_1 & 2d_2 - d_1 \\ 2d_2 - d_1 & 0 \end{bmatrix} \nless 0.
\]
Applications of diagonal stability

- Dynamic models for biochemical reactions
- Systems theory
- Population dynamics
- Communication networks
- Mathematical economics

Applications of simultaneous diagonal stability

- Large-scale dynamic systems
- Interconnected time-varying and switched systems
A Necessary and Sufficient Condition Based on Schur Complement

- We shall denote \( \langle k \rangle = \{1, 2, \ldots, k\} \). For \( A \in \mathbb{R}^{n \times n} \), let \( A[\alpha, \beta] \) be the submatrix of \( A \) whose rows and columns are indexed by \( \alpha, \beta \subseteq \langle n \rangle \), respectively, and let \( A[\alpha] = A[\alpha, \alpha] \).

- The Schur complement of \( A[\alpha] \) in \( A \) is defined as
  \[
  A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c],
  \]
  where \( \alpha^c = \langle n \rangle \setminus \alpha \), provided that \( A[\alpha] \) is nonsingular.
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The Schur complement of $A[\alpha]$ in $A$ is defined as

$$A/A[\alpha] = A[\alpha^c] - A[\alpha^c, \alpha]A[\alpha]^{-1}A[\alpha, \alpha^c],$$

where $\alpha^c = \langle n \rangle \backslash \alpha$, provided that $A[\alpha]$ is nonsingular.

Consider, for example, the partitioned matrix $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$, where $A[\alpha] = B$, $A[\alpha^c] = E$, $A[\alpha^c, \alpha] = D$, and $A[\alpha, \alpha^c] = C$. Then,

$$A/A[\alpha] = E - DB^{-1}C.$$
Theorem (Redheffer, 1985)

Let \( A \in \mathbb{R}^{n \times n} \) be a nonsingular matrix with \( A[\{n\}] > 0 \) and \( \alpha = \langle n - 1 \rangle \). Then, \( A \) is diagonally stable if and only if \( A[\alpha] \) and \( A^{-1}[\alpha] \) have a common diagonal solution.

Theorem (Shorten and Narendra, 2009)

Let \( A \in \mathbb{R}^{n \times n} \) be partitioned as \( A = [\hat{A} \quad p \quad q] \), where \( \hat{A} \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( r > 0 \). Then, \( A \) is diagonally stable if and only if \( \hat{A} \) and \( \hat{A} - pq^T r \) have a common diagonal solution.
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**Theorem (Shorten and Narendra, 2009)**

Let \( A \in \mathbb{R}^{n \times n} \) be partitioned as \( A = \begin{bmatrix} \hat{A} & p \\ q^T & r \end{bmatrix} \), where \( \hat{A} \in \mathbb{R}^{(n-1) \times (n-1)} \) and \( r > 0 \). Then, \( A \) is diagonally stable if and only if \( \hat{A} \) and \( \hat{A} - \frac{pq^T}{r} \) have a common diagonal solution.

\[
\left( \hat{A} - \frac{pq^T}{r} \right)^{-1} = A^{-1}[\langle n - 1 \rangle]
\]
Theorem

Let \( A \in \mathbb{R}^{n \times n} \) be partitioned as \( A = \begin{bmatrix} \hat{A} & p \\ q^T & r \end{bmatrix} \), where \( \hat{A} \in \mathbb{R}^{(n-1) \times (n-1)} \).

Then, \( A \) is diagonally stable with a diagonal solution \( D = \begin{bmatrix} \hat{D} \\ x \end{bmatrix} \), where \( \hat{D} \in \mathbb{R}^{(n-1) \times (n-1)} \), if and only if the following are true:

(i) \( r > 0 \).

(ii) \( \hat{A} \) and the Schur complement \( A/A[\{n\}] = \hat{A} - \frac{pq^T}{r} \) share a common diagonal solution \( \hat{D} \).
Lemma (Horn and Johnson, 1985)

Suppose that $B \in \mathbb{R}^{n \times n}$ is a symmetric matrix and $\alpha \subset \langle n \rangle$. Then, $B \succ 0$ if and only if

$$B[\alpha] \succ 0$$

and

$$B/B[\alpha] \succ 0.$$ 

Slyvester’s Determinant Theorem

Let $U \in \mathbb{R}^{n \times m}$ and $V \in \mathbb{R}^{m \times n}$. Then

$$\det(I_n + UV) = \det(I_m + VU),$$

where $I_k$ is the $k \times k$ identity matrix.
Proof of Theorem: We need to justify that, for some $x > 0$,

$$B = DA + A^T D = \begin{bmatrix} \hat{D} \hat{A} + \hat{A}^T \hat{D} & \hat{D} p + x q \\ p^T \hat{D} + x q^T & 2x r \end{bmatrix} \succ 0.$$
Proof of Theorem: We need to justify that, for some $x > 0$, 

$$B = DA + A^T D = \begin{bmatrix} \hat{D}A + \hat{A}^T \hat{D} & \hat{D}p + xq \\ p^T \hat{D} + xq^T & 2xr \end{bmatrix} \succeq 0.$$ 

This, by lemma with $\alpha = \langle n - 1 \rangle$ and $M = B[\alpha] = \hat{D}A + \hat{A}^T \hat{D} \succ 0$, is equivalent to that for some $x > 0$, 

$$f(x) = \frac{B}{B[\alpha]} = 2xr - (p^T \hat{D} + xq^T)M^{-1}(\hat{D}p + xq) > 0. \quad (6)$$
Proof of Theorem: We need to justify that, for some $x > 0$,

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From (6), $f(x) \leq 0$ whenever $x \leq 0$. On the other hand,

$$f(x) = -x^2 q^T M^{-1} q - 2x(q^T M^{-1} \hat{D} p - r) - p^T \hat{D} M^{-1} \hat{D} p.$$
Proof of Theorem: We need to justify that, for some $x > 0$,

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This, by lemma with $\alpha = \langle n - 1 \rangle$ and $M = B[\alpha] = \hat{D} \hat{A} + \hat{A}^T \hat{D} \succ 0$, is equivalent to that for some $x > 0$,

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It suffices to show

$$\Delta = (q^T M^{-1} \hat{D} p - r)^2 - (q^T M^{-1} q)(p^T \hat{D} M^{-1} \hat{D} p) > 0.$$
Hence, we calculate
\[
\Delta = \det \begin{bmatrix}
-r + q^T M^{-1} \hat{D} p & q^T M^{-1} q \\
p^T \hat{D} M^{-1} \hat{D} p & -r + p^T \hat{D} M^{-1} q
\end{bmatrix}
\]
Hence, we calculate
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p^T \hat{D} M^{-1} \hat{D} p & -r + p^T \hat{D} M^{-1} q
\end{bmatrix}
= r^2 \det \left( l_2 - \begin{bmatrix}
r^{-1} & r^{-1} \\
r^{-1} & r^{-1}
\end{bmatrix} \begin{bmatrix}
q^T \\
p^T \hat{D}
\end{bmatrix} \begin{bmatrix}
M^{-1} \hat{D} p & M^{-1} q
\end{bmatrix} \right).
\]
Hence, we calculate

\[
\Delta = \text{det} \begin{bmatrix}
-r + q^T M^{-1} \hat{D} p & q^T M^{-1} q \\
-p^T \hat{D} M^{-1} \hat{D} p & -r + p^T \hat{D} M^{-1} q
\end{bmatrix}
\]

\[
= r^2 \text{det} \left( I_2 - \begin{bmatrix} r^{-1} \\
r^{-1} \end{bmatrix} \begin{bmatrix} q^T \\
p^T \hat{D} \end{bmatrix} \begin{bmatrix} M^{-1} \hat{D} p & M^{-1} q \end{bmatrix} \right).
\]

By Sylvester’s determinant theorem, we have

\[
\Delta = r^2 \text{det} \left( I_{n-1} - \begin{bmatrix} M^{-1} \hat{D} p & M^{-1} q \end{bmatrix} \begin{bmatrix} r^{-1} \\
r^{-1} \end{bmatrix} \begin{bmatrix} q^T \\
p^T \hat{D} \end{bmatrix} \right).
\]
Hence, we calculate
\[
\Delta = \det \begin{bmatrix}
- r + q^T M^{-1} \hat{D} p & q^T M^{-1} q \\
p^T \hat{D} M^{-1} \hat{D} p & - r + p^T \hat{D} M^{-1} q
\end{bmatrix}
= r^2 \det \left( I_2 - \begin{bmatrix}
r^{-1} & r^{-1} \\
 & r^{-1}
\end{bmatrix} \begin{bmatrix} q^T \\
p^T \hat{D}
\end{bmatrix} \begin{bmatrix}
M^{-1} \hat{D} p & M^{-1} q
\end{bmatrix} \right).
\]

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M^{-1} \hat{D} p & M^{-1} q
\end{bmatrix} \begin{bmatrix} r^{-1} & r^{-1} \\
 & r^{-1}
\end{bmatrix} \begin{bmatrix} q^T \\
p^T \hat{D}
\end{bmatrix} \right).
\]

Continuing with the above, we finally arrive at
\[
\Delta = r^2 \det(M^{-1}) \det(\hat{D} S + S^T \hat{D}) > 0,
\]
where \( S = A/A[\{n\}] \).
We may specify all the feasible positive $D[\{n\}] = x$ values in a diagonal solution $D = \begin{bmatrix} \hat{D} \\ x \end{bmatrix}$ as follows:

- $x$ is in, but does not exceed, $0 \leq x_1 < x < x_2 \leq \infty$, where

$$x_1 = \frac{p^T \hat{D} M^{-1} \hat{D} p}{\sqrt{\Delta} - (q^T M^{-1} \hat{D} p - r)}$$

and

$$x_2 = \frac{\sqrt{\Delta} - (q^T M^{-1} \hat{D} p - r)}{q^T M^{-1} q},$$

with

$$M = \hat{D} \hat{A} + \hat{A}^T \hat{D}$$

and

$$\Delta = (q^T M^{-1} \hat{D} p - r)^2 - (q^T M^{-1} q)(p^T \hat{D} M^{-1} \hat{D} p).$$

In particular, when $q = 0$, $x_1 = \frac{p^T \hat{D} M^{-1} \hat{D} p}{2r}$ and $x_2 = \infty$. 
Corollary 1

Let $A \in \mathbb{R}^{n \times n}$ and $\alpha = \langle n \rangle \setminus \{k\}$ for some $1 \leq k \leq n$. Then, $A$ is diagonally stable matrix that has a diagonal solution $D$ with $D[\alpha] = \hat{D}$ and $D[\{k\}] = x$ if and only if the following are true:

(i) $A[\{k\}] > 0$.

(ii) $A[\alpha]$ and the Schur complement $A/A[\{k\}]$ share a common diagonal solution $\hat{D}$.

- The diagonal stability of a matrix $A$ is preserved under simultaneous row and column permutations on $A$.

- If a matrix $A$ is diagonally stable, then any Schur complement $A/A[\alpha]$ is also diagonally stable for any $\alpha \subseteq \langle n \rangle$. 
Corollary 2

Let $A^{(1)}, A^{(2)}, \ldots, A^{(m)} \in \mathbb{R}^{n \times n}$ be each partitioned as $A^{(k)} = \begin{bmatrix} \hat{A}^{(k)} & p^{(k)} \\ (q^{(k)})^T & r^{(k)} \end{bmatrix}$, where $\hat{A}^{(k)} \in \mathbb{R}^{(n-1) \times (n-1)}$. Then $A^{(1)}, A^{(2)}, \ldots, A^{(m)}$ have a common diagonal solution in the form $D = \begin{bmatrix} \hat{D} \\ x \end{bmatrix}$, with $\hat{D} \in \mathbb{R}^{(n-1) \times (n-1)}$, if and only if the following are true:

(i) $r^{(k)} > 0$, $k = 1, 2, \ldots, m$.

(ii) $\hat{A}^{(k)}$ and $A^{(k)} / A^{(k)}[\{n\}]$, $k = 1, 2, \ldots, m$, have $\hat{D}$ as a common diagonal solution.

(iii) $x_1 < x_2$, where $x_1 = \max_{1 \leq k \leq m} x^{(k)}_1$, $x_2 = \min_{1 \leq k \leq m} x^{(k)}_2$, and where for each $k$,

\[
x^{(k)}_1 = \frac{(p^{(k)})^T \hat{D}(M^{(k)})^{-1} \hat{D}p^{(k)}}{\sqrt{\Delta^{(k)}} - ((q^{(k)})^T (M^{(k)})^{-1} \hat{D}p^{(k)} - r^{(k)})}
\]

and

\[
x^{(k)}_2 = \frac{\sqrt{\Delta^{(k)}} - ((q^{(k)})^T (M^{(k)})^{-1} \hat{D}p^{(k)} - r^{(k)})}{(q^{(k)})^T (M^{(k)})^{-1} q^{(k)}}
\]

with

\[
M^{(k)} = \hat{D}\hat{A}^{(k)} + (\hat{A}^{(k)})^T \hat{D}.
\]
Corollary 3

For \( k = 1, 2, \ldots, m \), let \( A^{(k)} = [a_{i,j}^{(k)}] \in \mathbb{R}^{2 \times 2} \). Then, \( A^{(1)}, A^{(2)}, \ldots, A^{(m)} \) have a common diagonal solution \( D = \begin{bmatrix} 1 & x \end{bmatrix} \) if and only if the following hold:

(i) \( A^{(1)}, A^{(2)}, \ldots, A^{(m)} \) are all \( P \)-matrices.

(ii) \( x_1 < x_2 \), where \( x_1 = \max_{1 \leq k \leq m} x_1^{(k)} \), \( x_2 = \min_{1 \leq k \leq m} x_2^{(k)} \), and where for each \( k \), \( 0 \leq x_1^{(k)} < x_2^{(k)} \leq \infty \) are such that

\[
\begin{aligned}
    x_1^{(k)} &= \left( \frac{a_{1,2}^{(k)}}{\sqrt{a_{1,1}^{(k)} a_{2,2}^{(k)}} + \sqrt{\det(A^{(k)})}} \right)^2 \\
    x_2^{(k)} &= \left( \frac{\sqrt{a_{1,1}^{(k)} a_{2,2}^{(k)}} + \sqrt{\det(A^{(k)})}}{a_{2,1}^{(k)}} \right)^2.
\end{aligned}
\]
Example

\[
A_1 = \begin{bmatrix}
2 & 1 & -1 \\
-2 & 1 & -3 \\
-4 & 3 & 4
\end{bmatrix}, \quad
A_2 = \begin{bmatrix}
4 & 4 & -1 \\
-2 & 4 & 2 \\
0 & 3 & 2
\end{bmatrix}, \quad
A_3 = \begin{bmatrix}
1 & -3 & 2 \\
6 & 2 & -1 \\
-6 & -1 & 3
\end{bmatrix}.
\]
Example

\[ A_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & -3 \\ -4 & 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 4 & -1 \\ -2 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -3 & 2 \\ 6 & 2 & -1 \\ -6 & -1 & 3 \end{bmatrix}. \]

Taking \( \alpha = \langle 2 \rangle \), we obtain from Corollary 3 that \( A_1[\alpha], A_1/A_1[\alpha^c], A_2[\alpha], A_2/A_2[\alpha^c], A_3[\alpha], \) and \( A_3/A_3[\alpha^c] \) have a common diagonal solution

\( \hat{D} = \begin{bmatrix} 1 \\ x \end{bmatrix} \), where

\[
0.877 \approx \frac{121}{4(2 + \sqrt{15})^2} < x < \frac{(\sqrt{2} + 2\sqrt{5})^2}{36} \approx 0.962.
\]
Example

\[
A_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & -3 \\ -4 & 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 4 & -1 \\ -2 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -3 & 2 \\ 6 & 2 & -1 \\ -6 & -1 & 3 \end{bmatrix}.
\]

Taking \( \alpha = \langle 2 \rangle \), we obtain from Corollary 3 that \( A_1[\alpha] \), \( A_1/A_1[\alpha^c] \), \( A_2[\alpha] \), \( A_2/A_2[\alpha^c] \), \( A_3[\alpha] \), and \( A_3/A_3[\alpha^c] \) have a common diagonal solution

\[
\hat{D} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad \text{where} \quad 0.877 \approx \frac{121}{4(2 + \sqrt{15})^2} < x < \frac{(\sqrt{2} + 2\sqrt{5})^2}{36} \approx 0.962.
\]

If we choose, for example, \( x = 0.9 \) and assume that \( D = \begin{bmatrix} \hat{D} \\ y \end{bmatrix} \), then we can apply Corollary 2 on \( A_1 \), \( A_2 \), and \( A_3 \) to determine that

\[
0.604 \approx \frac{1026}{1393 + \sqrt{93649}} < y < \frac{1347 + 6\sqrt{6389}}{2570} \approx 0.71.
\]
Example

\[ A_1 = \begin{bmatrix} 2 & 1 & -1 \\ -2 & 1 & -3 \\ -4 & 3 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 4 & -1 \\ -2 & 4 & 2 \\ 0 & 3 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & -3 & 2 \\ 6 & 2 & -1 \\ -6 & -1 & 3 \end{bmatrix}. \]

Taking \( \alpha = \langle 2 \rangle \), we obtain from Corollary 3 that \( A_1[\alpha], A_1/A_1[\alpha^c], A_2[\alpha], A_2/A_2[\alpha^c], A_3[\alpha], \) and \( A_3/A_3[\alpha^c] \) have a common diagonal solution

\[ \hat{D} = \begin{bmatrix} 1 \\ x \end{bmatrix}, \text{ where } 0.877 \approx \frac{121}{4(2 + \sqrt{15})^2} < x < \frac{(\sqrt{2} + 2\sqrt{5})^2}{36} \approx 0.962. \]

If we choose, for example, \( x = 0.9 \) and assume that \( D = \begin{bmatrix} \hat{D} \\ y \end{bmatrix} \), then we can apply Corollary 2 on \( A_1, A_2, \) and \( A_3 \) to determine that

\[ 0.604 \approx \frac{1026}{1393 + \sqrt{93649}} < y < \frac{1347 + 6\sqrt{6389}}{2570} \approx 0.71. \]

Hence, given any \( y \) in the above range, \( A_1, A_2, \) and \( A_3 \) share a common diagonal solution in the form \( D = \begin{bmatrix} 1 \\ 0.9 \\ y \end{bmatrix}. \)
Figure 1: Change in the smallest eigenvalue of $Q_i = DA_i + A_i^T D$, $i=1,2,3$, depending on $y$, the last diagonal entry of $D$. 
### Theorem (Barker, Berman and Plemmons, 1978)

A matrix $A \in \mathbb{R}^{n \times n}$ is diagonally stable if and only if for every nonzero $X \succeq 0$, $AX$ has a positive diagonal entry.

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A New Characterization for Common Diagonal Solutions

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A matrix $A \in \mathbb{R}^{n \times n}$ is diagonally stable if and only if for every nonzero $X \succeq 0$, $AX$ has a positive diagonal entry.

**Theorem (Berman, Goldberg and Shorten, 2014)**

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$. Then, $\mathcal{A}$ has a common diagonal solution if and only if for any $X^{(k)} \succeq 0$, $k = 1, 2, \ldots, m$, not all of them zero, $\sum_{k=1}^{m} A^{(k)}X^{(k)}$ has a positive diagonal entry.
Theorem (Kraaijevanger, 1991)

The following statements are equivalent for a matrix $A \in \mathbb{R}^{n \times n}$:

(i) $A$ is diagonally stable.

(ii) $A \circ S$ is a $P$-matrix for all $S \succeq 0$ with diagonal entries all being 1.

(iii) $A$ has positive diagonal entries and $\det(A \circ S) > 0$ for all $S \succeq 0$ with diagonal entries all being 1.

- Hadamard product of two matrices $A = [a_{i,j}] \in \mathbb{R}^{n \times n}$ and $B = [b_{i,j}] \in \mathbb{R}^{n \times n}$ is the matrix $A \circ B = [a_{i,j}b_{i,j}] \in \mathbb{R}^{n \times n}$.

- A matrix $A$ is called a $P$-matrix ($P_0$-matrix) if all its principal minors are positive (nonnegative).
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- A matrix $A$ is called a $P$-matrix ($P_0$-matrix) if all its principal minors are positive (nonnegative).

- We shall extend Kraaijevanger’s result to a new characterization for a set of matrices to share a common diagonal solution.

- Accordingly, we shall extend $P$-matrices by introducing a new notion called $\mathcal{P}$-sets.
**Lemma (Fiedler and Ptak, 1962)**

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is a $P$-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, $x_i(Ax)_i > 0$ for some index $i$. 
Lemma (Fiedler and Ptak, 1962)

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is a $P$-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, $x_i(Ax)_i > 0$ for some index $i$.

Definition

Given $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$, we define $\mathcal{A}$ as a $\mathcal{P}$-set if for any $x^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, not all of them zero, there exists some index $i$ such that $\sum_{k=1}^{m} x_i^{(k)}(A^{(k)}x^{(k)})_i > 0$. If $A$ has a common diagonal solution, then it is a $\mathcal{P}$-set.
Lemma (Fiedler and Ptak, 1962)

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is a $P$-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, $x_i(Ax)_i > 0$ for some index $i$.

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Given $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$, we define $\mathcal{A}$ as a $\mathcal{P}$-set if for any $x^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, not all of them zero, there exists some index $i$ such that $\sum_{k=1}^{m} x^{(k)}_i (A^{(k)} x^{(k)})_i > 0$.

Theorem

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$. Then, $\mathcal{A}$ is a $\mathcal{P}$-set if and only if for any $x^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, not all of them zero,

$$\sum_{k=1}^{m} A^{(k)} x^{(k)} (x^{(k)})^T$$

has a positive diagonal entry.

- If $\mathcal{A}$ has a common diagonal solution, then it is a $\mathcal{P}$-set.
Main Theorem-1

Given \( \mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n} \), the following are equivalent:

(i) \( \mathcal{A} \) has a common diagonal solution.

(ii) \( \{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \ldots, A^{(m)} \circ S^{(m)}\} \) has a common diagonal solution for all \( S^{(k)} \succeq 0, \ k = 1, 2, \ldots, m \), each with diagonal entries being all 1.

(iii) \( \{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \ldots, A^{(m)} \circ S^{(m)}\} \) is a \( \mathcal{P} \)-set for all \( S^{(k)} \succeq 0, \ k = 1, 2, \ldots, m \), each with diagonal entries being all 1.

Outline of the proof:

(i) \( \Rightarrow \) (ii): Let \( A^{(k)}D + D(A^{(k)})^T \succ 0 \) for all \( k \). Then
\[
(A^{(k)} \circ S^{(k)})D + D(A^{(k)} \circ S^{(k)})^T = (A^{(k)}D + DA^{(k)}) \circ S^{(k)} \succ 0. \tag{7}
\]

(ii) \( \Rightarrow \) (iii): \( \mathcal{P} \)-set property is a necessary condition of simultaneous diagonal stability.

(iii) \( \Rightarrow \) (i): Any \( X^{(k)} \succeq 0 \) can be expressed in the form \( X^{(k)} = D^{(k)} S^{(k)} D^{(k)} \) for some \( S^{(k)} \succeq 0 \), whose diagonal entries all equal to 1, where \( D^{(k)} \) is the diagonal matrix with \( D^{(k)}_{i,i} = \sqrt{X^{(k)}_{i,i}}, \ i = 1, 2, \ldots, n \). Let \( y^{(k)} \in \mathbb{R}^n \) be such that \( y^{(k)}_{i} = D^{(k)}_{i,i} \) for all \( i \). Then,
\[
\left[ \sum_{k=1}^{m} (A^{(k)} \circ S^{(k)}) y^{(k)} (y^{(k)})^T \right]_{j,j} = \left[ \sum_{k=1}^{m} A^{(k)} X^{(k)} \right]_{j,j} \tag{8}
\]
Theorem

Assume $A = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$. Then, $A$ is a $P$-set if and only if

$$\sum_{k=1}^{m} A^{(k)} \circ y^{(k)}(y^{(k)})^T$$

is a $P$-matrix for any $y^{(k)} \in \mathbb{R}^n$, $k = 1, 2, \ldots, m$, such that for each index $i$, $y_i^{(k)} \neq 0$ for some $1 \leq k \leq m$. 


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Theorem

Assume \( \mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n} \). Then, \( \mathcal{A} \) is a \( \mathcal{P} \)-set if and only if

\[
\sum_{k=1}^{m} A^{(k)} \circ y^{(k)}(y^{(k)})^T \text{ is a } P\text{-matrix for any } y^{(k)} \in \mathbb{R}^n, k = 1, 2, \ldots, m, \text{ such that for each index } i, \ y_i^{(k)} \neq 0 \text{ for some } 1 \leq k \leq m.
\]

Main Theorem-2

Given \( \mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n} \), the following are equivalent:

(i) \( \mathcal{A} \) has a common diagonal solution.

(ii) \[
\sum_{k=1}^{m} A^{(k)} \circ S^{(k)} \text{ is a } P\text{-matrix for all } S^{(k)} \succeq 0, k = 1, 2, \ldots, m, \text{ provided that for any index } 1 \leq i \leq n, S_{i,i}^{(k)} = 1 \text{ for some } 1 \leq k \leq m.
\]

(iii) \( A_{i,i}^{(k)} > 0 \) for \( i = 1, 2, \ldots, n \) and \( k = 1, 2, \ldots, m \), and

\[
\det \left( \sum_{k=1}^{m} A^{(k)} \circ S^{(k)} \right) > 0 \text{ for all } S^{(k)} \succeq 0, k = 1, 2, \ldots, m, \text{ provided that for any index } 1 \leq i \leq n, S_{i,i}^{(k)} = 1 \text{ for some } 1 \leq k \leq m.
\]
Consider a partition $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ of $\langle n \rangle$, where $\langle n \rangle = \alpha_1 \cup \cdots \cup \alpha_p$ with these $\alpha_k$ being nonempty and mutually exclusive. When $p = 1$, we simply write $\alpha = \langle n \rangle$.

A block diagonal matrix with diagonal blocks indexed by $\alpha_1, \ldots, \alpha_p$ is said to be $\alpha$-diagonal.

A diagonal matrix $D \in \mathbb{R}^{n \times n}$ is called $\alpha$-scalar if, for each $1 \leq k \leq p$, $D[\alpha_k]$ is a scalar multiple of the identity matrix of the same size.

\[ A = \begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_p \end{bmatrix}, \quad D = \begin{bmatrix} c_1 I_1 & & \\ & c_2 I_2 & \\ & & \ddots \\ & & & c_p I_p \end{bmatrix} \]

$A_k \in \mathbb{R}^{n_k \times n_k}$ for $n_k = |\alpha_k|$  

$l_k \in \mathbb{R}^{n_k \times n_k}$ for $n_k = |\alpha_k|$
Definition (Hershkowitz and Mashal, 1998)

Let \( \alpha = \{\alpha_1, \ldots, \alpha_p\} \) be a of \( \langle n \rangle \). A matrix \( A \in \mathbb{R}^{n \times n} \) is said to be \( H(\alpha) \)-stable (-semistable) if \( AH \) is stable (semistable) for any positive definite \( \alpha \)-diagonal matrix \( H \).

- In particular, \( H(\langle n \rangle) \)-stability is also called \( H \)-stability.
**Definition (Hershkowitz and Mashal, 1998)**

Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a of $\langle n \rangle$. A matrix $A \in \mathbb{R}^{n \times n}$ is said to be $H(\alpha)$-stable (-semistable) if $AH$ is stable (semistable) for any positive definite $\alpha$-diagonal matrix $H$.

- In particular, $H(\langle n \rangle)$-stability is also called $H$-stability.

**Definition (Hershkowitz and Mashal, 1998)**

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be Lyapunov $\alpha$-scalar stable (semistable) if there exists some positive definite $\alpha$-scalar matrix $D$ such that

$$AD + DA^T \succ 0 \quad (AD + DA^T \succeq 0).$$

- We shall abbreviate Lyapunov $\alpha$-scalar stability as $L(\alpha)$-stability and use the term $L$-stability when $\alpha = \langle n \rangle$. 
**Definition**

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be additive $D$-stable (-semistable) if $A + D$ is stable (semistable) for any nonnegative diagonal matrix $D$. Additive $D$-stability arises in diffusion models of biological systems after linearization at the equilibrium, and guarantees the asymptotic stability of the equilibrium. Additive $D$-stability has also found applications in neural networks, mathematical economics and mathematical ecology.

**Theorem**

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is additive $D$-stable if $A$ is stable and $L(\alpha)$-semistable for some partition $\alpha$ of $\langle n \rangle$. 

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Definition

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Definition
Let $\alpha$ be a partition of $\langle n \rangle$. Then, a matrix $A \in \mathbb{R}^{n \times n}$ is said to be additive $H(\alpha)$-stable (-semistable) if $A + H$ is stable (semistable) for any positive semidefinite $\alpha$-diagonal matrix $H$.

- When $\alpha = \{\{1\}, \ldots, \{n\}\}$, additive $H(\alpha)$-stability is same as additive $D$-stability. When $\alpha = \langle n \rangle$, we also use the term additive $H$-stability in place of $H(\langle n \rangle)$-stability.
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- Additive $H(\alpha)$-stability can be interpreted as a criterion for the equilibrium of the following general diffusion problem to be asymptotically stable:

$$
\frac{\partial u}{\partial t} = \sum_{i,j=1}^{n} h_{i,j} \frac{\partial^2 u}{\partial x_i \partial x_j} + f(u),
$$

where $H = [h_{i,j}] \succeq 0$. Additive $H(\alpha)$-stability arises if, in addition, $H$ has an $\alpha$-diagonal structure.
**Lemma (Fiedler and Ptak, 1966)**

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is a $P_0$-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, there exists an index $i$ such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$. 

---

**Definition**

Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\langle n \rangle$. A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is said to be a $P_0(\alpha)$-matrix if for any nonzero $x \in \mathbb{R}^n$, there exists some $1 \leq k \leq p$ such that $(Ax)_{\alpha_k} \neq 0$ and $x_{\alpha_k}^T(Ax)_{\alpha_k} \geq 0$.

For given $\beta \subseteq \langle n \rangle$, $x_{\beta}$ is the subvector of $x$ indexed by $\beta$.

When $\alpha = \{\{1\}, \ldots, \{n\}\}$, a $P_0(\alpha)$-matrix is a nonsingular $P_0$-matrix. When $\alpha = \langle n \rangle$, a $P_0(\alpha)$-matrix is a nonsingular positive semidefinite, but not necessarily symmetric, matrix.

The notion of $P_0(\alpha)$-matrices bridges such general positive semidefinite matrices and nonsingular $P_0$-matrices.
**Lemma (Fiedler and Ptak, 1966)**

Let \( A \in \mathbb{R}^{n \times n} \). Then, \( A \) is a \( P_0 \)-matrix if and only if for any nonzero \( x \in \mathbb{R}^n \), there exists an index \( i \) such that \( x_i \neq 0 \) and \( x_i(Ax)_i \geq 0 \).

**Definition**

Let \( \alpha = \{\alpha_1, \ldots, \alpha_p\} \) be a partition of \( \langle n \rangle \). A nonsingular matrix \( A \in \mathbb{R}^{n \times n} \) is said to be a \( P_0(\alpha) \)-matrix if for any nonzero \( x \in \mathbb{R}^n \), there exists some \( 1 \leq k \leq p \) such that \( (Ax)[\alpha_k] \neq 0 \) and \( x[\alpha_k]^T(Ax)[\alpha_k] \geq 0 \).

- For given \( \beta \subseteq \langle n \rangle \), \( x[\beta] \) is the subvector of \( x \) indexed by \( \beta \).
**Lemma (Fiedler and Ptak, 1966)**

Let $A \in \mathbb{R}^{n \times n}$. Then, $A$ is a $P_0$-matrix if and only if for any nonzero $x \in \mathbb{R}^n$, there exists an index $i$ such that $x_i \neq 0$ and $x_i(Ax)_i \geq 0$.

**Definition**

Let $\alpha = \{\alpha_1, \ldots, \alpha_p\}$ be a partition of $\langle n \rangle$. A nonsingular matrix $A \in \mathbb{R}^{n \times n}$ is said to be a $P_0(\alpha)$-matrix if for any nonzero $x \in \mathbb{R}^n$, there exists some $1 \leq k \leq p$ such that $(Ax)_{[\alpha_k]} \neq 0$ and $x_{[\alpha_k]}^T(Ax)_{[\alpha_k]} \geq 0$.

- For given $\beta \subseteq \langle n \rangle$, $x_{[\beta]}$ is the subvector of $x$ indexed by $\beta$.
- When $\alpha = \{\{1\}, \ldots, \{n\}\}$, a $P_0(\alpha)$-matrix is a nonsingular $P_0$-matrix. When $\alpha = \langle n \rangle$, a $P_0(\alpha)$-matrix is a nonsingular positive semidefinite, but not necessarily symmetric, matrix.
- The notion of $P_0(\alpha)$-matrices bridges such general positive semidefinite matrices and nonsingular $P_0$-matrices.
Main Results

Regular matrix stability

\[ A \text{ is diagonally stable.} \]
\[ \Downarrow \]
\[ A \text{ is additive } D\text{-stable.} \]
\[ \Downarrow \]
\[ A \text{ is nonsingular } P_0\text{-matrix.} \]
\[ \leftrightarrow \]
\[ A + D \text{ is nonsingular for any nonnegative diagonal matrix } D \]

\[ A \text{ is } L(\alpha)\text{-stable.} \]
\[ \Downarrow \]
\[ A \text{ is additive } H(\alpha)\text{-stable.} \]
\[ \Downarrow \]
\[ A \text{ is a } P_0(\alpha)\text{-matrix.} \]
\[ \leftrightarrow \]
\[ A + H \text{ is nonsingular for any } \alpha\text{-diagonal } H \succeq 0. \]

A one way implication means that the converse does not hold in general.
Main Results

\[ A \text{ is } H\text{-stable.} \]

\[ \Downarrow \]

\[ A \text{ is additive } H\text{-stable.} \]

\[ \Downarrow \]

\[ A \text{ is stable and } A + bb^T \text{ is nonsingular for any } b \in \mathbb{R}^n. \]

\[ \Downarrow \]

\[ A \text{ is stable and } A + A^T \succeq 0. \]

\[ \Downarrow \]

\[ A \text{ is stable and a } P_0(\langle n \rangle)\text{-matrix.} \]

\[ A \text{ is } H\text{-stable.} \]

\[ \Downarrow \]

\[ A + P \text{ is } H\text{-stable for any } P \succeq 0. \]

\[ A \text{ is } H\text{-stable.} \]

\[ \Downarrow \]

\[ A + K \text{ is } L\text{-stable for any } K \succ 0. \]

- A one way implication means that the converse does not hold in general.
\[ A \in \mathbb{R}^{n \times n} \] is a nonsingular \( P_0 \)-matrix if and only if \( A + D \) is nonsingular for any nonnegative diagonal matrix \( D \) if and only if \( A \) is nonsingular and \( A + D \) is nonsingular for any positive diagonal matrix \( D \).

**Conjecture 1**

Let \( \alpha \) be a partition of \( \langle n \rangle \) and \( A \in \mathbb{R}^{n \times n} \). Then, the following are equivalent:

(i) \( A \) is a \( P_0(\alpha) \)-matrix.

(ii) \( A + H \) is nonsingular for every positive semidefinite \( \alpha \)-diagonal matrix \( H \).

(iii) \( A \) is nonsingular and \( A + H \) is nonsingular for every positive definite \( \alpha \)-diagonal matrix \( H \).

**Conjecture 2**

Let \( \alpha \) be a partition of \( \langle n \rangle \) and let \( A \in \mathbb{R}^{n \times n} \). If \( A \) is \( H(\alpha) \)-stable, then \( A \) is a \( P_0(\alpha) \)-matrix.
On-going work

**Theorem (Hershkowitz and Mashal, 1998)**

Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, the following statements are equivalent for a matrix $A$:

(i) $A$ is $L(\alpha)$-stable.

(ii) For every nonzero $X \succeq 0$, there exists some $1 \leq k \leq r$ such that $\text{tr}((AX)[\alpha_k]) > 0$.

**Theorem (Hershkowitz and Mashal, 1998)**

Let $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, the following statements are equivalent for a matrix $A$:

(i) $A$ is $L(\alpha)$-stable.

(ii) $A \circ S$ is a $P(\alpha)$-matrix for all $S \succeq 0$ with diagonal entries all being 1.

- $A \in \mathbb{R}^{n \times n}$ is said to be a $P(\alpha)$-matrix if for any nonzero $x \in \mathbb{R}^n$, there exists some $1 \leq k \leq r$ such that $x[\alpha_k]^T (Ax)[\alpha_k] > 0$. 
Definition

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$ and $\alpha$ be a partition of $\langle n \rangle$. If there exists some positive definite $\alpha$-scalar matrix $D$ such that

$$DA^{(j)} + (A^{(j)})^T D \succ 0, \ j = 1, 2, \ldots, m,$$

then $D$ is called a common $L(\alpha)$-solution for the matrix set $\mathcal{A}$. The existence of such a $D$ is interpreted as the simultaneous $L(\alpha)$-stability of all the matrices in $\mathcal{A}$.

Definition

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then we define $\mathcal{A}$ as a $P(\alpha)$-set if for any vector $x^{(j)} \in \mathbb{R}^n, \ j = 1, 2, \ldots, m$, not all of them zero, there exists $1 \leq k \leq r$ such that

$$\sum_{j=1}^{m} x^{(j)}[\alpha_k]^T (A^{(j)} x^{(j)})[\alpha_k] > 0.$$
Theorem

Let $\mathcal{A} = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, $\mathcal{A}$ has a common $L(\alpha)$-solution if and only if for any $X^{(j)} \succeq 0$, $j = 1, \ldots, m$, not all of them zero, there exist $1 \leq k \leq r$ such that

$$\text{tr} \left( \sum_{j=1}^{m} (A^{(j)} X^{(j)}) [\alpha_k] \right) > 0.$$
Theorem

Let $A = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, $A$ has a common $L(\alpha)$-solution if and only if for any $X^{(j)} \succeq 0$, $j = 1, \ldots, m$, not all of them zero, there exist $1 \leq k \leq r$ such that

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Let $A = \{A^{(1)}, A^{(2)}, \ldots, A^{(m)}\} \subset \mathbb{R}^{n \times n}$ and $\alpha = \{\alpha_1, \ldots, \alpha_r\}$ be a partition of $\langle n \rangle$. Then, the following are equivalent:

(i) $A$ has a common $L(\alpha)$-solution.

(ii) $\{A^{(1)} \circ S^{(1)}, A^{(2)} \circ S^{(2)}, \ldots, A^{(m)} \circ S^{(m)}\}$ is a $P(\alpha)$-set for all $S^{(j)} \succeq 0$, $j = 1, 2, \ldots, m$, with all diagonal entries are equal to 1.

(iii) $\sum_{j=1}^{m} A^{(j)} \circ S^{(j)}$ is a $P(\alpha)$-matrix for all $S^{(j)} \succeq 0$, $j = 1, 2, \ldots, m$, provided that for any index $1 \leq i \leq n$, $S^{(j)}_{i,i} = 1$ for some $1 \leq j \leq m$. 
Future works

- Explicit algebraic conditions for the diagonal stability and the simultaneous diagonal stability of higher order matrices.

- Extension of simultaneous diagonal stability problem to the simultaneous $L(\alpha)$-stability case.

- Characterization of $H(\alpha)$-stability and additive $H(\alpha)$-stability.

- Stability properties of structured matrices.
Sinc matrix $I^{(-1)} = S + \frac{1}{2}ee^T$, where $e \in \mathbb{R}^n$ is the vector of all ones and

$$S = \begin{bmatrix}
    s_0 & -s_1 & -s_2 & \ldots & -s_{n-1} \\
    s_1 & s_0 & -s_1 & \ldots & -s_{n-2} \\
    s_2 & s_1 & s_0 & \ldots & -s_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    s_{n-1} & s_{n-2} & s_{n-3} & \ldots & s_0
\end{bmatrix},$$

and $s_k = \int_0^k \text{sinc}(x)dx$, where $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}, \forall x \neq 0$, while $\text{sinc}(0) = 1$.

- $S$ is a skew-symmetric and Toeplitz matrix.
- A recent result confirmed that the Sinc matrix $I^{(-1)}$ is stable, but it is still unknown yet as to whether this matrix has $D$-stability, a problem key to various applications of Sinc methods.
References


- A new characterization of simultaneous Lyapunov diagonal stability via Hadamard products with Dr. Jianhong Xu, submitted to Linear and Multilinear Algebra.


THANK YOU