[20] 1. Compute the following limits. If the limit does not exist, explain why. Do not use L'Hopital's rule.

a) \( \lim_{x \to \infty} \frac{3 + 4x^2}{6x - x^2} \)

\[
\lim_{x \to \infty} \frac{\frac{3}{x^2} + \frac{4x}{x^2}}{\frac{6}{x^2} - \frac{x}{x^2}} = \lim_{x \to \infty} \frac{3 + 4x}{6x - x^2} = \lim_{x \to \infty} \frac{\frac{3}{x^2} + \frac{4}{x}}{\frac{6}{x^2} - \frac{1}{x}} = \frac{4}{1} = 4
\]

b) \( \lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 8x + 12} = \frac{x + 4 - 8}{x - 6 + 12} = \frac{0}{0} \)

\[
\lim_{x \to 2} \frac{(x + 4)(x - 2)}{(x - 2)(x - 6)} = \frac{2 + 4}{2 - 6} = \frac{6}{-4} = \frac{-3}{2}
\]

c) \( \lim_{x \to 0} \frac{2x}{\tan 3x} = \frac{2(0)}{\tan(3 \cdot 0)} = \frac{0}{0} \)

\[
\lim_{x \to 0} \frac{2x}{\sin 3x} = \lim_{x \to 0} \frac{2x \cos 3x}{\sin 3x} = \lim_{x \to 0} \frac{2x}{3x} = \frac{2(0)}{3} = 0
\]

d) \( \lim_{x \to -1} \frac{x + 2}{(x + 1)^2} = \frac{-1 + 2}{(-1 + 1)^2} = \frac{1}{0^2} \)

\[
\lim_{x \to -1} \frac{x + 2}{(x + 1)^2} = \frac{1}{0^2} = 1
\]

Check if the limit from the left and right is the same. If not, \( \lim_{x \to -1} \frac{x + 2}{(x + 1)^2} = \frac{2}{3} \)

So, \( \lim_{x \to -1} \frac{x + 2}{(x + 1)^2} = +\infty \)
2. Suppose \( f(x) = \frac{1}{x^2 + 2} \). Find \( f'(x) \) from the DEFINITION of the derivative.

\[
\lim_{h \to 0} \frac{\frac{1}{(x+h)^2+2} - \frac{1}{x^2+2}}{h} = \lim_{h \to 0} \frac{x^2+2 - (x+h)^2-2}{h(x^2+2)(x+h)^2+2} = \lim_{h \to 0} \frac{x^2+2 - x^2 - 2xh - h^2 - 2}{h(x^2+2)(x+h)^2+2} = \lim_{h \to 0} \frac{x^2 - 2x - h}{h(x^2+2)(x+h)^2+2}
\]

3. Find \( f'(x) \) for the following functions. You need not simplify your answers.

a) \( f(x) = 2 \tan 3x - 4e^{2x} \)

\[
f'(x) = 2 \cdot \sec^2(3x) \cdot 3 - 4e^{2x} \cdot 2
\]

b) \( f(x) = \sin^2 x + \sin x^2 = (\sin x)^2 + \sin(x^2) \)

\[
f'(x) = 2 \sin x \cos x + \cos(x^2) \cdot 2x
\]
c) \( f(x) = \frac{x^2 + 3x}{4 + \cos 2x} \)

\[
f'(x) = \frac{(2x+3)(4+\cos 2x) - (-\sin(2x)) \cdot 2(x^2+3x)}{(4+\cos(2x))^2}
\]

d) \( f(x) = (1 + \sin x)^{2x} \)

\[
\ln y = 2x \ln(1 + \sin x)
\]
\[
\frac{1}{y} \cdot y' = 2 \ln(1 + \sin x) + \frac{1}{1 + \sin x} \cdot (\cos x \cdot 2x)
\]

e) \( f(x) = \int_{3}^{x^2} \frac{dt}{1 + t^3} \)

\[
\frac{d}{dx} \int_{3}^{x^2} \frac{dt}{1 + t^3} = \frac{1}{1 + (x^2)^3} \cdot 2x = \frac{2x}{1 + x^6}
\]

f) \( f(x) = x^2 \ln(1 + \sec 3x) \)

\[
f'(x) = 2x \ln(1 + \sec 3x) + \frac{1}{1 + \sec 3x} \cdot \sec(3x) \tan(3x) \cdot 3 \cdot x^2
\]

$$\frac{dy}{dx} e^x + e^y \frac{dy}{dx} = (1 \cdot e^y + e^y \frac{dy}{dx}) = 2x$$

$$\frac{dy}{dx} e^x + e^y \frac{dy}{dx} - e^y - xe^y = 2x$$

$$\frac{dy}{dx} e^x - xe^y \frac{dy}{dx} = 2x - e^y + ey$$

$$\frac{dy}{dx} = \frac{2x - e^y + ey}{e^x - xe^y}$$

[10] 5. Find an equation of the tangent line to $f(x) = x \ln x$ at $x = 2$.

$$f' = 1 \cdot \ln x + \frac{1}{x} \cdot x$$

$$X = 2$$

$$y = 2 \ln 2$$

$$m = 1 \cdot \ln 2 + 1$$

$$y - 2 \ln 2 = (\ln 2 + 1)(x - 2)$$

[8] 6. Find the absolute maximum and absolute minimum of $f(x) = \frac{x^2 - 4}{x^2 + 4}$ on $[-4, 4]$.

$$f' = \frac{2x(x^2 + 4) - 2x(x^2 - 4)}{(x^2 + 4)^2} = \frac{2x^3 + 8x - 2x^3 + 8x}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2}$$

$$\frac{16x}{(x^2 + 4)^2} = 0$$

$$16x = 0$$

$$x = 0$$

$$\text{Undefined nowhere since } x^2 + 4 \neq 0$$

$$f(-4) = \frac{16 - 4}{16 + 4} = \frac{12}{20} = \frac{3}{5}$$

$$f(0) = \frac{-4}{4} = -1$$

$$f(4) = \frac{12}{20} = \frac{3}{5}$$
7. If a snowball melts so that its surface area decreases at a rate of 1 cm²/min, find the rate at which the diameter decreases when the diameter is 10 cm.

\[
\frac{dSA}{dt} = -1 \text{ cm}^2/\text{min}
\]

\[
\frac{dd}{dt} = ? \text{ when } d = 10
\]

\[
SA = 4\pi r^2 = 4\pi \left(\frac{d}{2}\right)^2 = 4\pi \frac{d^2}{4}
\]

\[
\frac{dSA}{dt} = 2\pi d \cdot \frac{dd}{dt} = \pi d^2
\]

\[
-1 = 2\pi (10) \frac{dd}{dt} \Rightarrow \frac{dd}{dt} = -\frac{1}{20} \text{ cm/min}
\]

8. Graph the function which satisfies the following conditions.

\[
\lim_{x \to -\infty} f(x) = 0, \quad \lim_{x \to -4^-} f(x) = 0, \quad \lim_{x \to -4^+} f(x) = \infty, \quad \lim_{x \to 0^-} f(x) = \infty, \quad \lim_{x \to 0^+} f(x) = -\infty
\]

\[
f' > 0 \text{ on } (-2, 0) \cup (0, \infty);
\]

\[
f' < 0 \text{ on } (-\infty, -2)
\]

\[
f'' > 0 \text{ on } (-4, 0);
\]

\[
f'' < 0 \text{ on } (-\infty, -4) \cup (0, \infty)
\]
9. A box with a square base and open top must have a volume of 32,000 cm$^3$. Find the dimensions of the box that minimizes the amount of material used.

$$M = x^2 + xy + xy \quad \text{Bottom}$$
$$M = x^2 + 4xy \quad \text{Left + Front + Back}$$

$$M = x^2 + 4x \left( \frac{32000}{x^2} \right) = x^2 + \frac{128000}{x}$$

$$M' = 2x - \frac{128000}{x^2} = 0 \quad \Rightarrow \quad 2x^3 - 128000 = 0$$

$$x^3 = 64000 \quad x = 40$$

$$y = \frac{32000}{40^2} = 20$$

10. Suppose that the DERIVATIVE $f'$ of a function $f$ has the graph

This graph is not the graph of the function. It is the graph of the derivative of $f$.

a) Find the intervals where $f$ is increasing/decreasing.

Increasing $(-\infty, -1) \cup (1, 3)$

Decreasing $(-1, 1) \cup (3, \infty)$

b) Find the intervals where $f$ is concave up/down.

Concave up $(0, 2)$

Concave down $(-\infty, 0) \cup (2, \infty)$
11. Evaluate the following indefinite integrals.

a) \[ \int (x + 2)(x + 3)dx = \int x^3 + 5x^2 + 6x + C \]

b) \[ \int (\tan x - 3 \csc^2 x)dx \]

\[ = -\ln |\cos x| + 3 \cot x + C \]

\[ \tan x = \frac{\sin x}{\cos x} \]
\[ \tan x = \frac{\sin x}{\cos x} \]
\[ u = \cos x \]
\[ du = -\sin x dx \]
\[ -\int \frac{u}{u} du \]
\[ = -\ln |u| + C \]
\[ = -\ln |\cos x| + C \]

c) \[ \int \frac{dx}{x \ln x} \]

\[ = \int \frac{1}{x \ln x} dx \]
\[ = \int \frac{1}{u} du = \ln |u| + C \]
\[ = \ln |\ln x| + C \]

d) \[ \int xe^{-x^2}dx \]

\[ u = -x^2 \]
\[ du = -2x dx \]
\[ \frac{du}{-2} = x dx \]
\[ \int e^u du = \frac{1}{2} e^u + C \]
\[ = \frac{1}{2} e^{-x^2} + C \]
12. Find the following definite integrals.

a) \( \int_0^3 x \sqrt{x + 1} \, dx \)

\( u = x + 1 \Rightarrow x = u - 1 \)

\( du = dx \)

\[ \int_1^4 (u-1)u^{1/2} \, du = \int_1^4 (u^{3/2} - u^{1/2}) \, du = \left[ \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} \right]_1^4 \]

\[ = \frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2} - \left( \frac{2}{5}(1)^{5/2} - \frac{2}{3}(1)^{3/2} \right) \]

\[ = \frac{16}{5} - \frac{16}{3} + \frac{2}{3} \]

\[ = \frac{16}{15} \]

b) \( \int_0^{\pi/8} (\sin 4x + 3 \cos 2x) \, dx \)

\[ = \left[ -\cos(4x) + 3 \sin(2x) \right]_0^{\pi/8} \]

\[ = \frac{3}{5} - \frac{3}{2} + \frac{2}{2} \]

\[ = \frac{64}{5} - \frac{16}{3} + \frac{2}{3} \]

\[ = \frac{64}{15} \]

\[ = \frac{16}{15} \]
13. Let \( R \) be the region bounded by \( y = e^x \) and \( y = 1 \) and \( x = 2 \). Find the following.

(a) Area of \( R \).

\[
\begin{align*}
\int_0^2 e^x - 1 \, dx &= e^x - x \bigg|_0^2 \\
&= e^2 - 2 - (e^0 - 0) = e^2 - 2 - 1
\end{align*}
\]

(b) Volume of the solid obtained by rotating \( R \) about the line \( y \)-axis. SETUP ONLY.

\[
\int_0^2 2\pi x(e^x - 1) \, dx
\]

(c) Volume of the solid obtained by rotating \( R \) about the line \( y = -1 \). SETUP ONLY.