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# Math 575 Matrix Computations Qualifying Exam

Fall 2020

*Instructions*: Show all necessary work for full credit. Quote theorems, if needed, clearly and concisely. No calculators, computers, cell phones, books, or notes are allowed. Arrange your answers in order, use this exam as the cover page, write your name on top, and staple them together.

1. Let $\|\cdot\|$ be a matrix norm on $\mathbb{C}^{n\times n}$, which is also submultiplicative, and let $P \in \mathbb{C}^{n\times n}$ be a nonsingular matrix. For any $A \in \mathbb{C}^{n\times n}$, define $\|A\|_P = \|P^{-1}AP\|$.
   (a) Show that $\|\cdot\|_P$ is also a matrix norm on $\mathbb{C}^{n\times n}$.
   (b) Show that $\|\cdot\|_P$ is submultiplicative too.
2. Let $v \in \mathbb{C}^n$ be such that $\|v\|_2 = 1$.
   (a) State the Householder reflector $F$ that is constructed with $v$.
   (b) Show that $F = F^* = F^{-1}$.
   (c) Is $F$ diagonalizable? Explain.
3. Let $A \in \mathbb{C}^{m\times n}$, where $m \geqslant n$.
   (a) State the definition of a reduced singular value decomposition of $A$.
   (b) Show that there exist $W \in \mathbb{C}^{m\times n}$ with orthonormal columns and positive semidefinite $P \in \mathbb{C}^{n\times n}$ such that $A = WP$.
   (c) Is it true that $\operatorname{rank}(P) = \operatorname{rank}(A)$? Explain.
4. Denote by $\|\cdot\|$ both a vector norm on $\mathbb{C}^m$ and its induced matrix norm on $\mathbb{C}^{m\times m}$.
   (a) Show that if $B \in \mathbb{C}^{m\times m}$ is such that $\|B\| < 1$, then $I + B$ is nonsingular.
   (b) Under the same assumption as in part (a), show
   $$\|(I+B)^{-1}\| \leqslant \frac{1}{1-\|B\|}.$$

(c) Consider the linear system $Ax = b$, where $A \in \mathbb{C}^{m\times m}$ is nonsingular and $b \in \mathbb{C}^m$. Suppose that $A$ and $b$ are perturbed to $A + \delta A$ and $b + \delta b$, respectively, where $\delta A$ satisfies $\|\delta A\| < 1/\|A^{-1}\|$. Show

$$\frac{\|\delta x\|}{\|x\|} \leqslant \frac{\kappa(A)}{1 - \kappa(A)\frac{\|\delta A\|}{\|A\|}} \left( \frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|} \right).$$

5. Let $A \in \mathbb{R}^{m\times n}$, where $m \geqslant n$. Consider the following algorithm on $A$, where

$$\operatorname{sign}(y) = \begin{cases} 1, & y \geqslant 0, \\ -1, & y < 0, \end{cases}$$

and $e_1$ is the first column of $I$, an identity matrix whose size can be determined by the context:

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for  j = 1 : n
     x = A_{j:m,j}
     v_j = sign(x_1)||x||_2 e_1 + x
     v_j = v_j/||v_j||_2
     A_{j:m,j:n} = A_{j:m,j:n} - (2v_j)(v_j^* A_{j:m,j:n})
end
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(a) Find the asymptotic operation count for the above algorithm. You may use the fact

$$\sum_{k=1}^{n} k^2 = \frac{2n^3 + 3n^2 + n}{6}.$$

(b) Explain the potential problem that may occur if $v_j = \operatorname{sign}(x_1)\|x\|_2 e_1 + x$ is replaced with $v_j = \|x\|_2 e_1 + x$.

6. (a) Let $\tilde{f}(x)$ be an algorithm for a problem $f(x)$. State the definitions of stability and backward stability of $\tilde{f}$.

(b) For $x \in \mathbb{C}$, $f(x) = \dfrac{1}{1+x^2}$ is computed as $\tilde{f}(x) = 1 \oplus \left\{1 \oplus \left[\mathrm{fl}(x) \otimes \mathrm{fl}(x)\right]\right\}$. Determine whether this algorithm is backward stable, stable but not backward stable, or unstable.

*That's All, Folks!*