1. Let $p$ be a prime number and let $G$ be a finite group with $|G| \geq 1$. Suppose that any element of $G$ is of order $p^k$ for some integer $k$. Show that the center $Z(G)$ of $G$ contains at least one non-identity element.

2. Let $p \in \mathbb{N}$ be a prime number. Let $G$ be a finite group of order $p$. Find the cardinality of the group $\text{Aut}(G)$ and explain completely your answer.

3. Let $H, K$ be two groups. Given a semi-direct product $G = H \rtimes_{\varphi} K$ and a group homomorphism $\varphi : K \rightarrow \text{Aut}(H)$, show that $K \triangleleft G$ if and only if $G$ is isomorphic to the direct product $H \times K$.

4. Let $G$ be a group with property that each conjugacy class in $G$ has size at most two. Prove that $G' \leq Z(G)$. Here, $G'$ denotes the commutator subgroup.

5. Show that if $\sigma \in S_n$ is an $(n-1)$-cycle, where $n \geq 3$, then $C(\sigma) = \langle \sigma \rangle$.

6. Prove that any finite group $G$ of order $224 = 2^5 \cdot 7$ has a subgroup of order 28.

7. Consider $R = \mathbb{Z}[\sqrt{-5}]$ and a principal ideal $(2 + \sqrt{-5})$ in $R$. Show that there is no principal ideal $I$ such that 

$$(2 + \sqrt{-5}) \subset I \subsetneq R.$$
8. Let $R$ be a commutative ring with identity. For an ideal $I$ of $R$, define

$$V_I = \{ P \mid P \text{ is a prime ideal of } R \text{ and } I \subseteq P \}.$$ 

Let $I$ and $J$ be ideals of $R$. Prove

$$V_I \cup V_J = V_{IJ} = V_{I \cap J}$$

9. Decide if any the following rings are isomorphic. Justify your answer.

(a) $M_2(F)$, the ring of $2 \times 2$ matrices over the field with 4 elements $F = \mathbb{F}_2^2$.
(b) $M_2(\mathbb{Z}_4)$
(c) $\mathbb{Z}_{16} \oplus \mathbb{Z}_{16}$
(d) $\mathbb{Z}_4 \oplus \mathbb{Z}_{64}$

10. Let $F$ be a field, and consider the polynomial ring $R = F[x, y]$ in two indeterminates over $F$. Let $p(x)$ be an irreducible polynomial in $F[x]$. Let

$$E = F[x]/(p(x)).$$

(a) Prove that $R/p(x)R \cong E[y]$.
(b) Is $p(x)R$ a prime ideal of $R$? Is it a maximal ideal of $R$?