MATH 519 QUALIFYING EXAM

FALL 2017

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let p be a prime number and let G be a finite group with $|G| \gneqq 1$. Suppose that any element of G is of order p^k for some integer k. Show that the center $Z(G)$ of G contains at least one non-identity element.

2. Let $p \in \mathbb{N}$ be a prime number. Let G be a finite group of order p. Find the cardinality of the group $Aut(G)$ and explain completely your answer.

3. Let H, K be two groups. Given a semi-direct product $G = H \rtimes_\varphi K$ and a group homomorphism $\varphi : K \to Aut(H)$, show that $K \lhd G$ if and only if G is isomorphic to the direct product $H \times K$.

4. Let G be a group with property that each conjugacy class in G has size at most two. Prove that $G' \leq Z(G)$. Here, G' denotes the commutator subgroup.

5. Show that if $\sigma \in S_n$ is an $(n-1)$-cycle, where $n \geq 3$, then $C(\sigma) = \langle \sigma \rangle$.

6. Prove that any finite group G of order $224 = 2^5 \cdot 7$ has a subgroup of order 28.

7. Consider $R = \mathbb{Z}[\sqrt{-5}]$ and a principal ideal $(2 + \sqrt{-5})$ in R. Show that there is no principal ideal I such that

$$(2 + \sqrt{-5}) \subsetneqq I \subsetneqq R.$$

8. Let R be a commutative ring with identity. For an ideal I of R, define

$$V_I = \{P \mid P \text{ is a prime ideal of } R \text{ and } I \subseteq P\}.$$

Let I and J be ideals of R. Prove

$$V_I \cup V_J = V_{IJ} = V_{I \cap J}$$

9. Decide if any the following rings are isomorphic. Justify your answer.

(a) $M_2(F)$, the ring of 2×2 matrices over the field with 4 elements $F = \mathbb{F}_{2^2}$.
(b) $M_2(\mathbb{Z}_4)$
(c) $\mathbb{Z}_{16} \oplus \mathbb{Z}_{16}$
(d) $\mathbb{Z}_4 \oplus \mathbb{Z}_{64}$

10. Let F be a field, and consider the polynomial ring $R = F[x, y]$ in two indeterminates over F. Let $p(x)$ be an irreducible polynomial in $F[x]$. Let

$$E = F[x]/(p(x)).$$

(a) Prove that $R/p(x)R \cong E[y]$.
(b) Is $p(x)R$ a prime ideal of R? Is it a maximal ideal of R?