Math 501-Fall 2011 **PhD Qualifying Exam**

NAME: ID #

Closed book and notes. Show all of your work. Please turn in these sheets with your answers. Full credit (200 points) will only be given for rigorous mathematical justification.

λ denotes Lebesgue measure on $\mathbf{R}$; and λ^* is Lebesgue outer measure.

1.(30 points) Define what is meant by a *Lebesgue measurable set* in $\mathbf{R}$ and a *Lebesgue measurable function* $f : \mathbf{R} \to \mathbf{R}$.

(a)Let $f : \mathbf{R} \to [\mathbf{0}, \infty)$ be a function such that there are non-negative constants $c_n, n \geq 1$ and Lebesgue measurable sets $E_n, n \geq 1$, so that

$$f(x) = \sum_{n=1}^{\infty} c_n 1_{E_n}(x)$$

for all $x \in \mathbf{R}$. The characteristic (indicator) function of the set E_n is denoted by 1_{E_n}. Prove that f is Lebesgue measurable. Is f Borel measurable when each E_n is Borel measurable?

(b) Let $f_n : \mathbf{R} \to \mathbf{R}$, $n \geq 1$, be a sequence of real-valued Lebesgue measurable functions. Let E be the set of all $x \in \mathbf{R}$ such that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ converges in $\mathbf{R}$. Prove that E is Lebesgue measurable.

2.(40 points) Give an *outline* of the construction of *Lebesgue measure* λ on $\mathbf{R}$. Do not give proofs.

(a) Let $A \subset \mathbf{R}$ be a set of positive Lebesgue measure and suppose that $0 < \epsilon < \lambda(A)$. Show that there exists a Lebesgue measurable subset B of A such that $\lambda(B) = \epsilon$.

(b) Let $E \subset [0, 1]$ be a Lebesgue measurable set such that $\lambda(E) = 1$. Show that E is dense in $[0, 1]$.

(c) Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of subsets of $[0, 1]$ such that $\sum_{n=1}^{\infty} \lambda^*(A_n) < \infty$, where λ^* is Lebesgue outer measure. Show that the set

$$\{x \in [0, 1] : x \in A_n \text{ for infinitely many } n\}$$

has Lebesgue measure zero.

3.(30 points) Distinguish between a *Lebesgue-measurable* set and a *Borel measurable* set in $\mathbf{R}$. Let λ be Lebesgue measure on $\mathbf{R}$. Prove the following statements:

(a) The union of two Lebesgue measurable sets is Lebesgue measurable.

(b) If E is Lebesgue-measurable, then given $\epsilon > 0$, there is an open set U and a closed set F such that $F \subset E \subset U$ and

$$\lambda(U \backslash E) < \epsilon, \quad \lambda(E \backslash F) < \epsilon$$

(c) For every Lebesgue-measurable set E, there is a Borel measurable set B such that $E \subseteq B$ and $\lambda(E) = \lambda(B)$.

(d) For any real-valued function $f : \mathbf{R} \to \mathbf{R}$, the set of points at which f is discontinuous is a Borel set.

4.(30 points) Define what is meant by an extended-real-valued *(Lebesgue)-measurable* function $f : [a, b] \to \mathbf{R}^*$.

Let $f : [a, b] \to \mathbf{R}^*$ be a measurable function which is finite almost everywhere (with respect to Lebesgue measure), and let $\epsilon > 0$ be given. Prove the following two statements

(a) There exists $K > 0$ such that $|f(x)| \leq K$ for all x outside a set of measure less than ϵ.

(b) There is a simple function $\phi : [a, b] \to \mathbf{R}$ such that $|f(x) - \phi(x)| < \epsilon$ for all x outside a set of measure less than ϵ.

5.(30 points) Define *convergence in measure* (with respect to Lebesgue measure λ) for a sequence of measurable extended-real-valued functions $f_n : E \to \mathbf{R}^*,\ n \geq 1$. Prove the following version of *Fatou's Lemma:*

Let E be a subset of $\mathbf{R}$ of finite measure. Suppose $f_n : E \to \mathbf{R}^*,\ \mathbf{n} \geq \mathbf{1}$, is a sequence of non-negative measurable function such that $f_n \to f$ in measure on E. Then

$$\int_E f(x)\, d\lambda(x) \leq \liminf_{n \to \infty} \int f_n(x)\, d\lambda(x).$$

6.(30 points) Define what is meant by a *Lebesgue integrable function* $f : (0, \infty) \to \mathbf{R}$.

Consider the function $f : (0, \infty) \to \mathbf{R}$ defined by

$$f(x) := \frac{\sin x}{x}$$

for all $x > 0$. Show that the improper Riemann integral $(R) \int_0^\infty f(x)\, dx$ exists but the Lebesgue integral $\int_0^\infty f(x)\, d\lambda(x)$ does not exist. (*Hint: Prove that the series* $\sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} (\sin x/x)\, dx$ *is conditionally convergent.*)

7.(30 points) State *Lebesgue's Dominated Convergence Theorem* and the *Monotone Convergence Theorem.*

(a)Let $f_n : \mathbf{R} \to \mathbf{R}^{\geq \mathbf{0}}$ be a sequence of non-negative intergrable functions on $\mathbf{R}$ such that $f_n \to f$ almost everywhere, and f is an integrable function over $\mathbf{R}$ (with respect to Lebesgue measure λ). Suppose that

$$\lim_{n\to\infty} \int f_n(x)\, d\lambda(x) = \int f(x)\, d\lambda(x).$$

Show that for any Lebesgue measurable set E, one has

$$\lim_{n\to\infty} \int_E f_n(x)\, d\lambda(x) = \int_E f(x)\, d\lambda(x)$$

(b) Show that the following limit exists, evaluate it and justify all your calculations:

$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin(x/n)}{x(1+x^2)}\, d\lambda(x)$$

8.(30 points) Let $f : [0, 1] \to \mathbf{R}$ be a Lebesgue integrable function such that

$$\int_0^1 x^n f(x)\, d\lambda(x) = 0$$

for all integers $n \geq 0$. Show that $f = 0$ almost everywhere. Is your conclusion true if the above relation is to hold for all *even* integers:

$$\int_0^1 x^{2n} f(x)\, d\lambda(x) = 0, \quad n \geq 0?$$

Give reasons for your answer. (*Hint:* The algebra generated by $\{1, x^2\}$ contains the constant functions and separates points in $[0,1]$. Hence it is dense in $C([0,1], \mathbf{R})$ by the Stone-Weierstrass theorem .)

Suppose $f : [0,1] \to \mathbf{R}$ is Lebesgue integrable and satisfies

$$\int_0^1 f(x^{1/(2n+1)})\, d\lambda(x) = 0$$

for every integer $n \geq 0$. Show that $f = 0$ almost everywhere.

9.(30 points) For a sequence of functions $f_n : [0, \infty) \to \mathbf{R}, n \geq 1$, define what is meant by *convergence in measure*, *pointwise convergence*, *convergence almost everywhere* and *convergence in* L^1.

Consider the sequence $f_n : [0, \infty) \to \mathbf{R}, n \geq 1$, by

$$f_n := \sin(1/n) 1_{(0,n)}, \ n \geq 1,$$

where $1_{(0,n)}$ denotes the characteristic (or indicator) function of $(0, n)$. Decide whether the sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise, converges in measure, converges almost everywhere, or converges in L^1. Justify your answer in each case.

10.(30 points) Let $f : [0,1] \to \mathbf{R}$ be absolutely continuous. Prove the following statements:

(a) f is continuous on $[0,1]$.

(b) f is of bounded variation on $[0,1]$.

(c) If $E \subset [0,1]$ has Lebesgue measure zero, then $f(E)$ also has Lebesgue measure zero.