Ph.D. Qualifying Exam in MATH 530 (Topology)
August 2023

Instructions: Do as many of the ten problems below as you can. Please use a separate sheet of paper for each problem. If a problem takes more than one page, number the pages. Do not use the back of a page. Good luck!

1. (10 points; 2 points each)
   a) What is a Lindelöff space?
   b) Define local compactness.
   c) Define the compactification of a space.
   d) State the Urysohn Lemma.
   e) What is the quotient topology?
2. (10 points) Prove that the continuous image of a compact space is compact. Give an example showing that the continuous image of a closed set need not be closed.
3. (10 points) Construct an example showing that a quotient space of a Hausdorff space need not be Hausdorff.
4. (10 points) Prove the *Sequence Lemma*: Let X be a topological space. Let $A \subset X$. If there is a sequence of points in A converging to x, then $x \in \bar{A}$. The converse holds if X is a metric space.
5. (10 points) Show that $\mathbb{R}^\omega$ in the box topology is not metrizable. Hint: exploit the Sequence Lemma.
6. (10 points) Prove that a topological space X is locally path connected if and only if for every open set U of X, each component of U is open in X.
7. (10 points) Let X be a compact Hausdorff space. Prove that if X has no isolated points, then X is uncountable.
8. (10 points) Prove that the space $\mathbb{R}_l$ is normal. (Recall $\mathbb{R}_l$ has the topology generated by intervals of the form $[a, b)$.)
9. (10 points) Let D^2 be a closed disk. Show that there is no retraction from D^2 to its boundary. '
10. (10 points) a. What is the fundamental group of T^2 with one point deleted? b. What is the fundamental group of T^2 with two points deleted? (Hint: draw the torus as a square with opposite sides identified and then delete two points from the interior.)