Transforming Resources within a Quantum Resource Theory

Eric Chitambar

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Outline

- Survey the basic mathematical formalism of finite-dimensional quantum mechanics.

- Introduce the general framework of quantum resource theories.

- Describe the concepts of resource measures and resource transformations.

- Provide examples of resource transformations in entanglement and coherence theory.

Mathematical Structure of Quantum Mechanics

- Every quantum system is assigned a Hilbert space $\mathcal{H}$ called **state space**.
  - Finite-dimensional systems: $\mathcal{H} \approx \mathbb{C}^d$.

- Multiple quantum systems are combined by a tensor product of vector spaces.

$$\mathcal{H}^{AB\cdots N} = \mathcal{H}^A \otimes \mathcal{H}^B \otimes \cdots \otimes \mathcal{H}^N.$$

"Joint" state space

"Reduced" state spaces

Mathematical Structure of Quantum Mechanics

- Physical states of a system are described by **density operators** acting on the state space.

  - More precisely, for state space $\mathcal{H}$, let $\mathcal{L}(\mathcal{H})$ denote the set of linear operators acting on $\mathcal{H}$.

  - A density operator is any element $\rho \in \mathcal{L}(\mathcal{H})$ such that:

$$\left.\begin{array}{ll}\text{(i)} & \rho \geq 0, \\ \text{(ii)} & \mathrm{Tr}(\rho) = 1.\end{array}\right\}$$

Any operator satisfying these properties represents a valid state of the quantum system.

  - The set of density operators acting on $\mathcal{H}$ will be denoted $\mathcal{D}(\mathcal{H})$.

Example:

- For bipartite space $\mathcal{H}^A \otimes \mathcal{H}^B$, a density operator $\rho^{AB} = \sum_{i,j=1}^{d_A^2 d_B^2} c_{ij} M_i \otimes N_j$ satisfies

(i) $\forall |\Psi\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B$: $\sum_{i,j=1}^{d_A^2, d_B^2} c_{ij} \langle \Psi | M_i \otimes N_j | \Psi \rangle \geq 0$,

(ii) $\sum_{i,j=1}^{d_A^2, d_B^2} c_{ij} \mathrm{Tr}(M_i) \mathrm{Tr}(N_j) = 1$.

Mathematical Structure of Quantum Mechanics

- A **quantum operation** is any physical process that maps density matrices to density matrices.

  - It is a positive, trace-preserving map $\mathcal{E}$:

  $$\text{(i)} \quad \mathcal{E}(\rho) \geq 0,$$

  $$\text{(ii)} \quad \mathrm{Tr}(\mathcal{E}(\rho)) = 1.$$

  - Even stronger, these maps preserve density matrices when acting on just a subsystem:

  $$\text{(i)} \quad \mathcal{E}^A \otimes \mathcal{I}^B(\rho^{AB}) \geq 0,$$

  $$\text{(ii)} \quad \mathrm{Tr}(\mathcal{E}^A \otimes \mathcal{I}^B(\rho^{AB})) = 1.$$

  $\mathcal{I}^B$ is the identity map on system B.

    - These maps are called **completely positive**.

Every quantum operation is represented by a trace-preserving completely positive (CPTP) map.

Completely positive trace-preserving (CPTP) maps

- Important examples:

  - "Measurment Maps"

    - Any collection of positive operators $\{\Pi_k\}_{k=1}^K$ such that $\sum_{k=1}^K \Pi_k = 1^{\mathcal{H}}$ represents a quantum measurement.

    - In terms of a CPTP maps, measurements are described by a measurement map $\mathcal{M} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathbb{C}^K)$, such that

$$\mathcal{M}(\rho) = \sum_{k=1}^K \underbrace{\mathrm{Tr}(\rho\Pi_k)}\overbrace{|k\rangle\langle k|}$$

$|k\rangle\langle k|$ represents the state of some classical measurement device reading "outcome k".

$tr(\rho\Pi_k)$ represents the probability of obtaining "outcome k" when measuring the state ρ.

Completely positive trace-preserving (CPTP) maps

- Important examples:

  - "Partial Trace"

    - If $\rho^{AB} \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$ is a state for the joint state space $\mathcal{H}^A \otimes \mathcal{H}^B$, the **reduced state** of system A (or B) is described by the **partial trace** CPTP map $\mathrm{Tr}_B : \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B) \to \mathcal{D}(\mathcal{H}^A)$:

For $\rho^{AB} = \sum_{i,j=1}^{d_A^2 d_B^2} c_{ij} M_i \otimes N_j$,

$$\mathrm{Tr}_B(\rho^{AB}) := \sum_{i,j=1}^{d_A^2 d_B^2} c_{ij} M_i \mathrm{Tr}(N_j)$$

$$= \sum_{i=1}^{d_A^2} c_i' M_i =: \rho^A.$$

Quantum Resource Theories

- In many practical situations, the full aresnal of CPTP maps is not accessible to the experimenter.

  - Due to expermential limitations or environmental circumstances, the experimenter cannot perform the most general processes that nature allows.

  - For example, maybe a quantum computering engineer can only use certain types of gates in the design of a quantum computer circuit board.

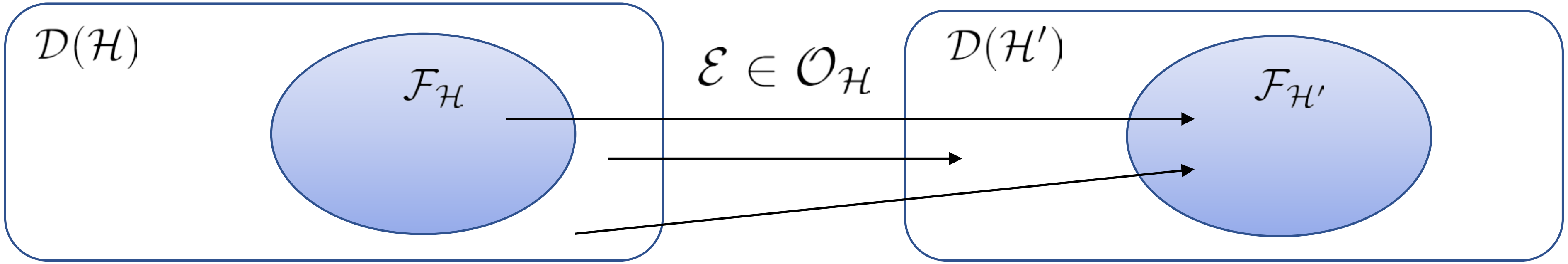
  - Or maybe two individulas want to communicate over some communication channel, but they only have a partially shared reference frame. Then there are a limited number of degrees of freedom to encode their messages.

- A quantum resource theory provides the mathematical framework to model the physics in these restricted settings.

Quantum Resource Theories

Definition: A **quantum resource theory** (QRT) is an assignment that associates every Hilbert space $\mathcal{H}$ with a tuple $(\mathcal{F}_{\mathcal{H}}, \mathcal{O}_{\mathcal{H}})$, where $\mathcal{F}_{\mathcal{H}}$ is a collection of density operators $\rho \in \mathcal{D}(\mathcal{H})$ and $\mathcal{O}_{\mathcal{H}}$ is a collection of CPTP maps acting on $\mathcal{D}(\mathcal{H})$, such that:

If $\mathcal{E} : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$ belongs to $\mathcal{O}_{\mathcal{H}}$,
then $\mathcal{E}(\rho) \in \mathcal{F}_{\mathcal{H}'}$ for all $\rho \in \mathcal{F}_{\mathcal{H}}$.



- The states in $\mathcal{F}_{\mathcal{H}}$ are called **free states**, and the operations in $\mathcal{O}_{\mathcal{H}}$ are called **free operations**.
- Any state $\rho \in \mathcal{D}(\mathcal{H}) \setminus \mathcal{F}_{\mathcal{H}}$ is called a **resource state**.

Entanglement: The Prototypical QRT

- The theory of quantum entanglement is the paradigmatic example of a quantum resource theory.

- The free states are convex combinations of tensor product states:

$$\rho^{AB} = \sum_k p_k \sigma_k^A \otimes \omega_k^B \qquad \text{where } \sum_k p_k = 1,$$

$$\sigma_k^A \in \mathcal{D}(\mathcal{H}^A),\ \omega_k^B \in \mathcal{D}(\mathcal{H}^B)$$

$$\sigma_k^A \otimes \omega_k^B \in \mathcal{D}(\mathcal{H}^A \otimes \mathcal{H}^B).$$

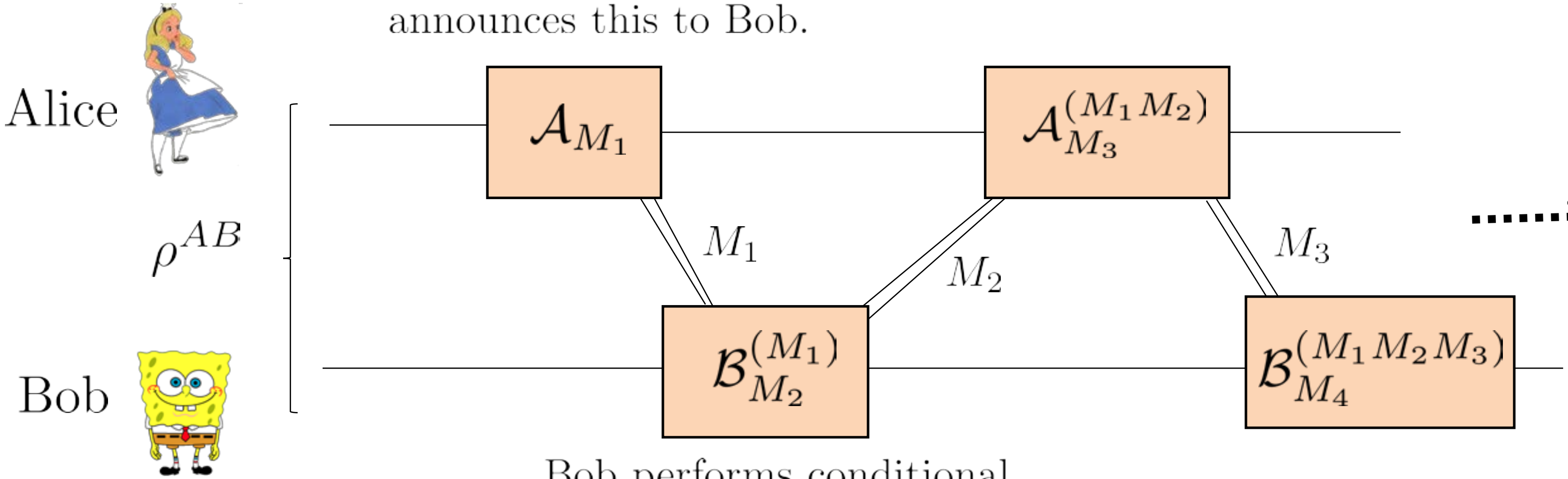
  - These are called **separable states**.

- Any state that cannot be expressed as a convex combination of product states is called **entangled**.

Entanglement: The Prototypical QRT

- The free operations consist of Local Operations and Classical Communication (LOCC).

Alice performs a quantum operation $\{\mathcal{A}_{M_1}\}_{M_1=1\ldots}$ on her subsystem generating measurement outcome M_1. She announces this to Bob.



$$\sigma^{AB} = \sum_{M_1 \cdots M_r = \vec{m}} \mathcal{A}_{\vec{m}}^{tot} \otimes \mathcal{B}_{\vec{m}}^{tot}(\rho^{AB})$$

Bob performs conditional quantum operation $\{\mathcal{B}_{M_2}^{(M_1)}\}_{M_2=1\ldots}$ on his subsystem generating measurement outcome M_2. He announces this Alice.

- The only information exchanged between Alice and Bob is classical.

- Entanglement cannot be generated in this way.

Three Other Examples of a QRT

- Asymmetry

  For some group G with unitary representation U_g.

  - Free states: $\rho = \tau(\rho) := \int_G dg U_g(\rho) U_g^\dagger$
  - Free operations: $[\mathcal{E}, \tau] = 0$

- Quantum Thermodynamics

  For a Hamiltonian H on system $\mathcal{H}$,

  - Free states:

    $$\gamma_T = e^{-H/kT} / \mathrm{Tr}(e^{-H/kT}).$$

    (Gibbs state at temperature T)

  - Free operations:

    $$\mathcal{E}(\rho) = \mathrm{Tr}_B[V(\rho \otimes \tilde{\gamma}_T)V^\dagger]$$

    where $\tilde{\gamma}_T$ is the Gibbs state at temperature T for some Hamiltonian $\tilde{H}$, and $[V, H \otimes 1 + 1 \otimes \tilde{H}] = 0$.

- Quantum Coherence

  For a fixed orthonormal basis $\mathcal{B} = \{|i\rangle\}_{i=1}^d$,

  - Free states: ρ diagonal in basis $\mathcal{B}$.
  - Free operations: Multiple proposals (stay tuned...)

Resource Measures and Monotones

- For a given quantum resource theory, we want to quantify the amount of resource in a given state.

- **Definition**: A function $\phi : \mathcal{D}(\rho) \to \mathbb{R}$ qualifies as a resource measure if:

(i) $\phi(\rho) \geq 0$ for all ρ, and $\underbrace{\phi(\rho) = 0 \text{ if } \rho \in \mathcal{F}}$.

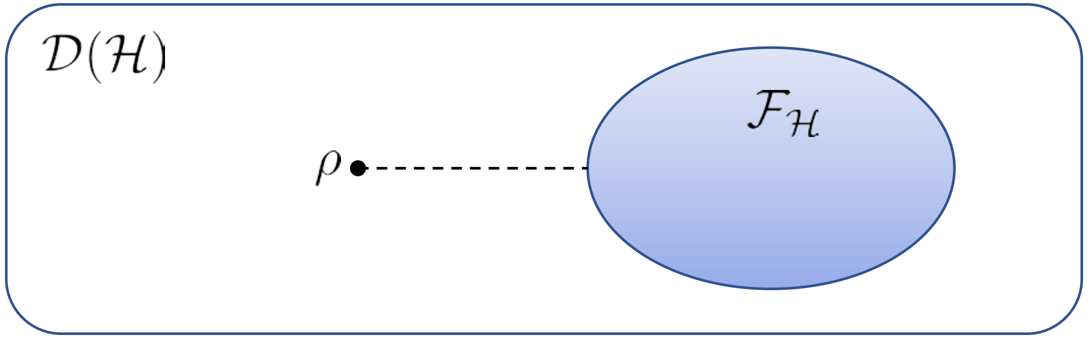
"Free states have no resource."

(ii) ϕ is monotonically decreasing under the free operations,

$$\underbrace{\phi(\rho) \geq \phi(\mathcal{E}(\rho))} \text{ for all } \rho \text{ and all } \mathcal{E} \in \mathcal{O}.$$

"The resource value cannot be increased by the free operations."

Distance-Based Measures



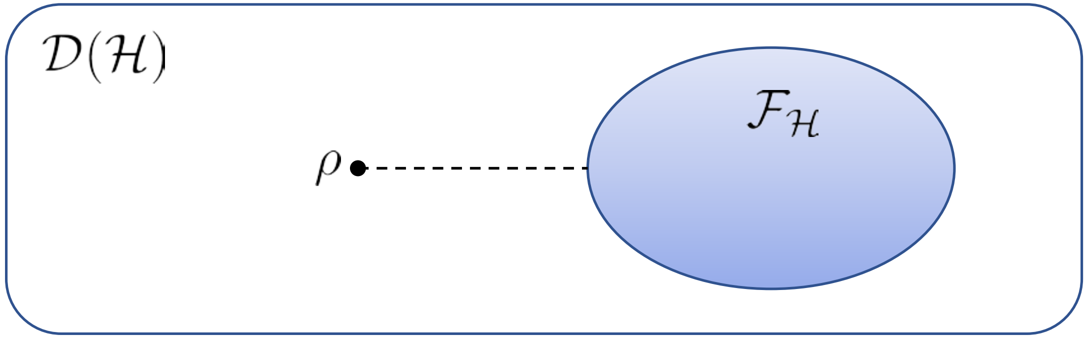
- The idea is to capture the amount of resource in ρ by "how far" it is from the set of free states.

  - Trace norm measures:

    For operator A, let $||A||_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ denote its "trace norm."

    It is known that $||A||_1 \geq ||\mathcal{E}(A)||_1$ for any CPTP map $\mathcal{E}$,
    i.e. the trace norm is contractive under CPTP maps.

Distance-Based Measures



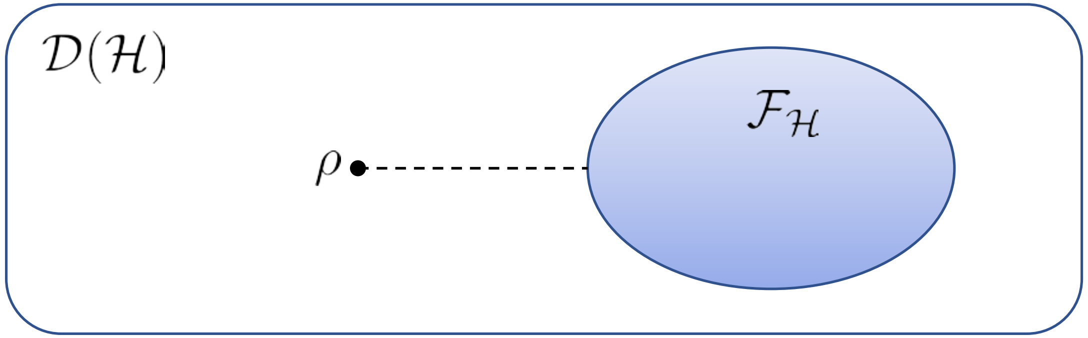
- Define the trace norm resource measure:

$$\mathcal{D}(\rho) = \inf_{\sigma \in \mathcal{F}} ||\rho - \sigma||_1.$$

- $\mathcal{D}(\rho) = 0$ if $\rho \in \mathcal{F}$.

- $\mathcal{D}(\rho) \geq \mathcal{D}(\mathcal{E}(\rho))$ for all ρ and all $\mathcal{E} \in \mathcal{O}$.

Distance-Based Measures



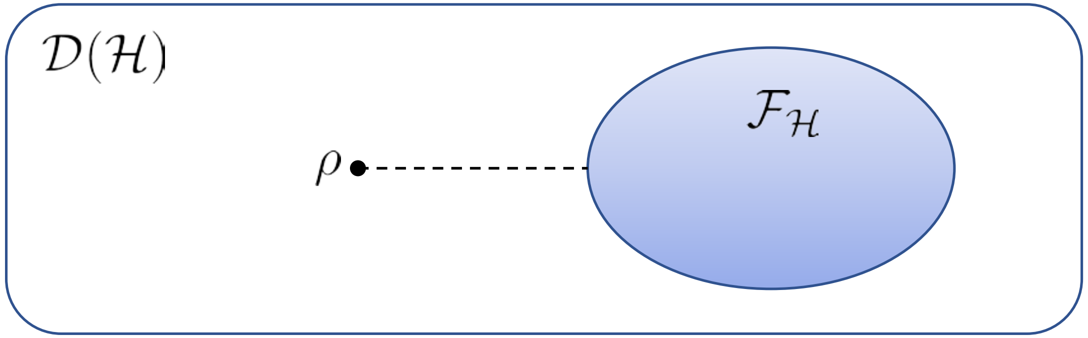
- Robustness-based measure:

For state ρ, define its **resource robustness**:

$$\mathcal{R}(\rho) = \inf\{\lambda : (1-\lambda)\sigma + \lambda\rho \in \mathcal{F},\ \sigma \in \mathcal{D}(\mathcal{H})\}.$$

- This quantifies the minimum amount of “noise” that must be mixed with ρ to destroy the resource.

Distance-Based Measures



- Entropic-based measure:

For state ρ, define its **relative entropy of resource**:

$$Rel(\rho) = \inf_{\sigma \in \mathcal{F}} S(\rho||\sigma),$$

where $S(\rho||\sigma) = \mathrm{Tr}[\rho \log(\rho/\sigma)]$ is the quantum relative entropy.

Resource Transformations in QRT

• One of the most important questions in the study of a QRT is whether one state ρ can be transformed into another σ using the free operations.

$$\rho \xrightarrow{\mathcal{E}\in\mathcal{O}} \sigma?$$

• Such a relationship provides an *operational* comparison between the resource content in two different states:

If $\rho \xrightarrow{\mathcal{E}\in\mathcal{O}} \sigma$, then ρ contains at least as much resource as σ, since any task using σ and the allowed operations can also be performed by ρ and the allowed operations.

Resource Transformation in Entanglement Theory

- Given two states ρ^{AB} and σ^{AB} when can $\rho^{AB} \to \sigma^{AB}$ by LOCC?
- In general this problem is unsolved. But for pure states, there is an easy answer.
- For a pure state $|\Psi\rangle^{AB}$, define its k^{th} Ky-Fan norm:

$$E_k(|\Psi\rangle) = \sum_{i=1}^{k} \lambda_i^{\downarrow}\left(\mathrm{Tr}_B|\Psi\rangle\langle\Psi|\right),$$

where $\lambda_i^{\downarrow}(X)$ denotes the i^{th} largest eigenvector of positive operator X.

- For every k, $E_k(|\Psi\rangle)$ is a resource monotone under LOCC.
- In fact, $|\Psi\rangle^{AB} \to |\Phi\rangle^{AB}$ by LOCC iff $E_k(|\Psi\rangle) \geq E_k(|\Phi\rangle)$ for all k.

Resource convertibility is determined entirely by a finite number of monotones!

Resource Transformation in Coherence Theory

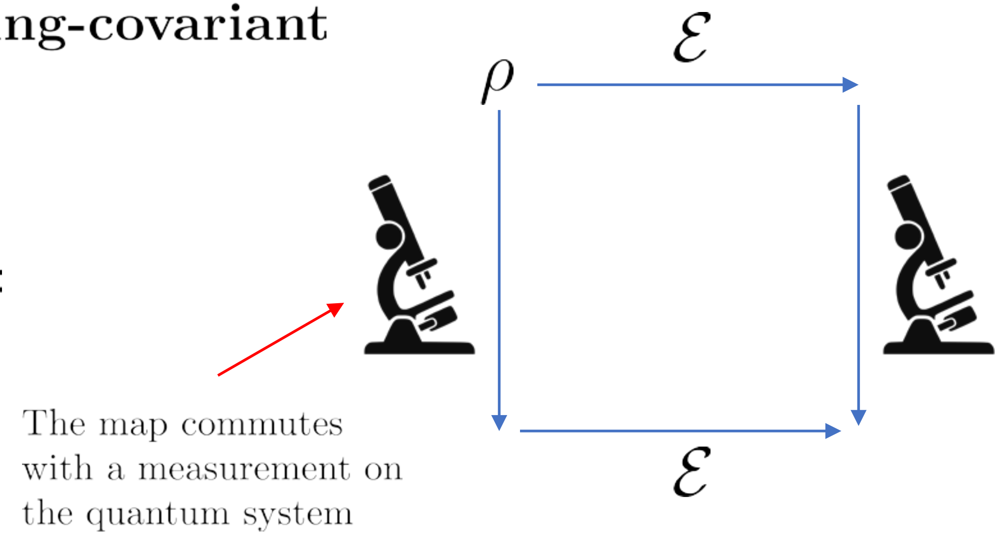
- In the quantum resource theory of coherence, free states are diagonal in some fixed basis $\mathcal{B} = \{|i\rangle\}_{i=1}^{d}$:

$$\rho = \sum_{i=1}^{d} p_i |i\rangle\langle i|.$$

- One type of free operations: dephasing-covariant.

Definition: A CPTP map $\mathcal{E}$ is called a **dephasing-covariant incoherent operation** (DIO) if

$$\mathcal{E}\left(\sum_{i=1}^{d} \langle i|\rho|i\rangle |i\rangle\langle i|\right) = \sum_{i=1}^{d} \langle i|\mathcal{E}(\rho)|i\rangle |i\rangle\langle i| :$$



Resource Transformations using DIO

- Dephasing covariant operations map the set of free states onto itself.
- What about for resource states, i.e. non-diagonal states?
- When is it possible to transform $\rho \to \sigma$ using a DIO map?
- In general the problem is not solved, however for two-level systems the answer is known.
- In two-level (qubit) systems: $$\rho = \begin{pmatrix} p & r \\ r & 1-p \end{pmatrix}.$$ W.l.o.g. assume $r \geq 0$.

Resource Transformations using DIO

- Recall the resource robustness:

$$\mathcal{R}(\rho) = \inf\{\lambda : (1-\lambda)\sigma + \lambda\rho \in \mathcal{F},\ \sigma \in \mathcal{D}(\mathcal{H})\}.$$

- It is a monotone under DIO.

- Introduce a new robustness measure:

$$\mathcal{R}_\Delta(\rho) = \inf\{\lambda : (1-\lambda)\sigma + \lambda\rho \in \mathcal{F},\ \langle i|\sigma-\rho|i\rangle = 0,\ \ \forall i = 1, \cdots d,\ \sigma \in \mathcal{D}(\mathcal{H})\}.$$

- This is also a monotone under DIO.

For $\rho = \begin{pmatrix} p & r \\ r & 1-p \end{pmatrix}$:

$$\mathcal{R}(\rho) = 2r,$$

$$\mathcal{R}_\Delta(\rho) = \frac{r}{\sqrt{p(1-p)}}.$$

$\rho \to \sigma$ under DIO iff
$\mathcal{R}(\rho) \geq \mathcal{R}(\sigma)$ and $\mathcal{R}_\Delta(\rho) \geq \mathcal{R}_\Delta(\sigma)$.

Conclusions

• Quantum resource theories provide a framework to isolate certain features of a quantum system, quantify it, and study its dynamical properties.

• A number of open problems involve understanding convertibility conditions $\rho \to \sigma$ in a given QRT.

- The same mathematical structures appear in many different resource theories, even those which seem otherwise unrelated.

• For example, by studying the convertibility of states under DIO (or similar operations) in coherence theory, we can learn something about the convertibility of states in entanglment theory.

Thank You