Qualifying Exam in Geometry and Topology
Spring 2011

1. Calculations

A. Let the following geometric objects be given in some neighborhood of a manifold M with coordinates $\{x,y,z\}$

$$f \in FM, \quad \alpha \in \Lambda^1 M, \quad \omega \in \Lambda^2 \mathrm{M}, \quad X, Y \in XM$$

$f = x^2 + yzt$
$\alpha = \mathrm{d}x + x\,\mathrm{d}y + z\,\mathrm{d}z + \mathrm{d}t$
$\mathrm{X} = x\,\partial_x$
$\mathrm{Y} = x^2\,\partial_y + \partial_x$
$\omega = (x^2 + y^2)\,\mathrm{d}x \wedge \mathrm{d}y + y\,\mathrm{d}z \wedge \mathrm{d}t$

Calculate the following

$$\text{rank of } \alpha, \quad \alpha\wedge\omega, \quad L_X f, \quad L_X \mathrm{Y}, \quad X \,\lrcorner\, \omega, \quad L_X\,\omega$$

B. On a manifold $M = \mathbf{R}^3$ with coordinates $\{x,y,z\}$ the standard inner product is given and a volume form is $\eta = (1+x^2)\,\mathrm{dx} \wedge \mathrm{dy} \wedge \mathrm{dz}$. Let also a "constant" vector field $\mathrm{X} = \partial_x$ be given and a biform $\omega = \mathrm{dx} \wedge \mathrm{dy}$, Calculate the following:

div X, curl X, $*\omega$ (the last is the Hodge star)

2. Definitions. Define (a) exterior derivative, (b) tangent vector, (c) Lie derivative of an exterior form.

3. Proofs (do 3)

a) State and prove Poincaré Lemma

(b) State and prove Stokes Theorem

(c) Show that the dual space of a finite-dimensional real linear space L has the same dimension as the space: dim L = dm L^*.

(d) Show that the two-dimensional sphere S^2 is a differentiable manifold

4. Simple questions. Explain what in mathematical folklore the following expressions mean:

a) sphere cannot be combed
b) exterior derivative d is a natural operation on a manifold

Give quick argument (one-line proof)

c) if each of the vector fields X and Y are symmetries of a dynamical system given by vector field Z, then the commutator [X,Y] is also its symmetry.
d) An integral of any exact bi-form on a two-dimensional torus is zero.

5. Derive the relation between Lie bracket of vector fields and exterior derivative of one-forms that starts

$$\mathrm{d}\alpha(X,Y) = -\,\alpha([X,Y]) + \ldots$$