MATH 519 QUALIFYING EXAM

SPRING 2017

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a group of order 8. Show that the factor group $G/Z(G)$ is abelian, where $Z(G)$ is the center of G.

2. Let H, K be two subgroups of a group G with $K \triangleleft G$. Suppose $|H|$ and $[G : K]$ are finite and that $|H|$ and $[G : K]$ are relatively prime. Show that $[H : H \cap K] = 1$.

3. Let p be a prime number. Show that any group G of order $3p$ is not simple.

4. A group is said to be *torsion-free* if no element other than the identity is of finite order. An abelian group G is said to be *of rank 1* if it has a non-torsion element (i.e., an element of infinite order) and if for any $x, y \in G$ we have integers m, n such that $mx + ny = 0$. Prove that any torsion-free abelian group of rank 1 is a subgroup of $\mathbb{Q}$.

5. Find three nonisomorphic groups of order 126. Show that they are not isomorphic to one another.

6. Show that for any groups A, B there exists a group C with group homomorphisms $f_A : A \to C$ and $f_B : B \to C$ such that for any group D with group homomorphisms $g_A : A \to D$ and $g_B : B \to D$, we have a group homomorphism $h : C \to D$ such that $g_A = h \circ f_A$ and $g_B = h \circ f_B$.

7. Let R be a commutative ring with $1 \neq 0$. Show that R is an integral domain if and only if the zero ideal $\langle 0 \rangle = \{0\}$ is a prime ideal.

8. Let R be a commutative ring with 1. Show that the set of all non-units in R equals the union of all maximal ideals in R.

9. Let V be the union of the two standard axes in the Euclidean plane $\mathbb{R}^2$. Let $R = \mathbb{R}[x, y]$, and let

$$I(V) = \{p(x, y) : p(a, b) = 0 \text{ for all } (a, b) \in V\}.$$

(1) Show that $I(V)$ is an ideal of R.
(2) Show that $I(V)$ is not prime.
(3) Give prime ideals J, K of R such that $I = J + K$.

10. Show that $\mathbb{Z}[x]$ is not a principal ideal domain.