# MATH 519 QUALIFYING EXAM

FALL 2018

*Directions.* Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let $G$ be a finite group. Denote by $Aut(G)$ the group of all automorphisms of $G$. Prove that $|Aut(G)| = 1$ implies $G = \{1\}$ or $G$ isomorphic to $\mathbb{Z}_2$.

2. Let $G$ be a group with $N \triangleleft G$. Show that there is *no surjective* group homomorphism $\varphi : G \to S_4$ such that $|S_4 : \varphi(N)| = 8$. Here, $\varphi(N)$ is the image of $N$ under $\varphi$.

3. Let $G$ be a group of oder $pq$, where $p, q$ are primes (not necessarily distinct). Show that $G$ is not simple.

4. Prove or disprove the following statement:

   (a) Let $x, y$ be two elements of finite order in a group. The element $xy$ has also finite order.

   (b) Let $H_i$ be normal subgroups of groups $G_i$ for $i = 1, 2$. If $H_1 \simeq H_2$ and if $G_1 \simeq G_2$, then $G_1/H_1 \simeq G_2/H_2$.

5. Let $G$ be a finite group, $K$ normal subgroup, and $P$ a Sylow $p$-subgroup of $G$. Prove that $P \cap K$ is a Sylow $p$-subgroup of $K$.

6. Consider $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, the quaternion group of order 8. (In $Q_8$, $i^2 = j^2 = k^2 = -1$, $ij = k$, $ji = -k$, $jk = i$, $kj = -i$, $ki = j$, $ik = -j$.)

   (a) Find all conjugacy classes and their sizes in $Q_8$.

   (b) Find $C_{Q_8}(k)$.

7. For any prime $p \geq 2$, show that there is always an irreducible polynomial in $\mathbb{Z}_p[x]$ whose degree is 2.

8. Let $R$ be a finite commutative ring with 1. Let $I \subset R$ be a prime ideal. Define $I[x]$ as the ideal of all polynomial in $R[x]$ whose coefficients lie in $I$, i.e.,

$$I[x] := \{a_0 + a_1 x + \cdots + a_n x^n : a_i \in I,\ n \in \mathbb{N} \cup \{0\}\}.$$

Show that $R[x]/I[x]$ is an Euclidean domain.

9. Let $R$ be a principal ideal domain. Prove that every proper ideal is a product $P_1 P_2 \cdots P_n$ of maximal ideals, which are uniquely determined up to order.

10. Let $R$ be a commutative ring with 1. Let us call an ideal $I$ of $R$ **irreducible** if it is NOT possible to write $I = I_1 \cap I_2$, where $I_1$ and $I_2$ are proper ideals of $R$ properly containing $I$.

(a) Let $x \in R$, $x \neq 0$. Show that there is an ideal $I_x$ of $R$ maximal with respect to the property that $x \notin I_x$.

(b) Show that the ideal $I_x$ from part (a) is irreducible.

(c) Show that every prime ideal $P$ of $R$ is irreducible.