MATH 519 QUALIFYING EXAM

SPRING 2018

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a finite group of order $p^2(p+2)$, where p is an odd prime number. Assume that G possesses at least two distinct Sylow p-subgroups. Show that there is a normal subgroup of G whose order is p.

2. Let n be a positive integer and let p be a prime. Consider a multiplicative group G defined as $\{0 < \overline{a} < p^n : \text{g.c.d.}(\overline{a}, p^n) = 1\}$. For any $m \in \mathbb{Z}$ with g.c.d.$(m, p^n) = 1$, we write $\overline{m}$ for the remainder after dividing m by p^n. Denote by $|m|$ the order of the element $\overline{m}$ in the group G. Show that $|p^{n-1} + p^{n-2} + \cdots + p + 1| = |1 - p|$.

3. Recall that a group G is *solvable* if there is a chain of subgroups

$$\{1\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_s = G$$

such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, $i = 0, 1, \ldots, s-1$.
Let $p \geq 5$ be a prime number. Show that any group G of order $3p^k$ with $k \in \mathbb{N}$ is solvable.

4. Classify all groups of order 2018, up to isomorphism. Note that $2018 = 2 \cdot 1009$ and 1009 is prime.

5. Let G be a group such that for any two nontrivial elements $a, b \in G$, there is an automorphism of G sending a to b.

(a) Prove that all nontrivial elements of G have the same order. Furthermore, if some nontrivial element of G has finite order, then there is a prime p such that all nontrivial elements of G have order p.

(b) Prove that if G is finite then G is abelian.

6. Let G be a finite group with commutator subgroup G'. Let N be the subgroup of G generated by the set

$$\{x^2 \mid x \in G\}.$$

Show that N is a normal subgroup of G such that N contains G'.

7. Given two distinct prime numbers p, q, consider two irreducible polynomials $a(x) = x^{p-1} + x^{p-2} + \cdots + x + 1, b(x) = x^{q-1} + x^{q-2} + \cdots + x + 1$ in $\mathbb{Q}[x]$. Define a ring homomorphism
$$\phi : \mathbb{Q}[x] \to \mathbb{Q}[x]/(a(x)) \times \mathbb{Q}[x]/(b(x))$$
by $\phi(f(x)) = (f(x) + (a(x)), f(x) + (b(x)))$. Show that ϕ is surjective.

8. Let D be an integer and let
$$S = \{\begin{pmatrix} a & b \\ Db & a \end{pmatrix} \mid a, b \in \mathbb{Z}\}.$$

(a) Prove that S is a subring of the ring $M_2(\mathbb{Z})$ of 2×2-matrices with coefficients in $\mathbb{Z}$.

(b) Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers. Decide if the map $\varphi : S \to \mathbb{Z}[i]$ defined by
$$\varphi(\begin{pmatrix} a & b \\ Db & a \end{pmatrix}) = a + bi$$
is a ring homomorphism.

9. Let p be a prime number and let R be the set of all rational numbers with denominator prime to p. Then R is a subring of $\mathbb{Q}$. Any element $x \in R$ can be written as $x = p^r a/b$, where $p \not| \, ab$ and $r \geq 0$.

(a) Prove that R is a principal ideal domain.

(b) Prove that R has a unique maximal ideal $M = (p)$. Prove that R/M is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

10. Let R be a commutative ring with unity $1 \neq 0$. The nilradical N of R is the ideal consisting of the nilpotent elements of R, that is,
$$N = \{x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N}\}.$$
It can also be characterized as the intersection of all the prime ideals of the ring. Hence,
$$N = \bigcap_P P,$$
where P runs over the set of all prime ideals of R. Show that the following are equivalent.

(a) R has exactly one prime ideal.

(b) R/N is a field.

(c) Every element of R is a unit or nilpotent.