Math 519 Qualifying Exam

Fall 2019

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a group of order $1215 = 3^5 \cdot 5$. Show that G is solvable. Recall that a group G is *solvable* if there is a chain of subgroups

$$\{1\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_s = G$$

such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, $i = 0, 1, \ldots, s-1$.

2. Let G be a finite group of order n and let p be a prime number with $p|n$. For any Sylow p-subgroup P, show that

$$N_G(N_G(N_G(P))) = N_G(P),$$

where $N_G(\star)$ is the normalizer of $\star$ in G.

3. Let G be a group of oder $p \cdot q^{p-1}$, where $p < q$ are two distinct odd primes. Show that G has a normal q-subgroup.

4. Let $A = \langle a \rangle$ and $B = \langle b | b^3 \rangle$, that is, A and B are cyclic, with A infinite and B of order 3. Let $P = \langle a, b | b^3 \rangle$. Show that there is not a homomorphism of groups $h : A \times B \to P$ that commutes with the maps

$$\begin{aligned} p_1 : P \to A \quad &\text{by} \quad p_1(a) = a \text{ and } p_1(b) = 1_A \\ p_2 : P \to B \quad &\text{by} \quad p_2(b) = b \text{ and } p_2(a) = 1_B \\ \pi_1 : A \times B \to A \quad &\text{by} \quad \pi_1(a) = a \text{ and } \pi_1(b) = 1_A \\ \pi_2 : A \times B \to B \quad &\text{by} \quad \pi_2(b) = b \text{ and } \pi_2(a) = 1_B \end{aligned}$$

5. A nontrivial Abelian group A (written additively) is called *divisible* if for each $a \in A$ and each nonzero integer k, there exists an element $x \in A$ such that $kx = a$.

   (a) Prove that the additive group of rational numbers, $\mathbb{Q}$, is divisible.

   (b) Prove that no finite Abelian group is divisible.

   (c) Prove that a quotient of a divisible Abelian group by any proper subgroup is divisible.

6. A sequence of groups and group homomorphisms

$$N \xrightarrow{\alpha} G \xrightarrow{\beta} Q$$

is said to be *exact at G* if and only if the image of α is equal to the kernel of β. Let 1 denote the trivial group. A *short exact sequence* is a sequence of groups and group homomorphisms

$$1 \to N \xrightarrow{\alpha} G \xrightarrow{\beta} Q \to 1$$

that is exact at N, G, and Q.

(a) Construct homomorphisms α and β so that

$$1 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}/6\mathbb{Z} \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 1$$

is a short exact sequence.

(b) Let S_3 denote the symmetric group on 3 elements, and construct homomorphisms α and β such that

$$1 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\alpha} S_3 \xrightarrow{\beta} \mathbb{Z}/2\mathbb{Z} \to 1$$

is a short exact sequence.

7. Prove or disprove the following statement:

(1) Let $G = H \rtimes_{\varphi} K$ be a semi-direct product of two groups H, K with an homomorphism $\varphi : K \to Aut(H)$. Then, we have $K \lhd G$ if and only if φ is trivial.

(2) Let I, J be ideals of a commutative ring R. If $I + J = R$, then $IJ = I \cap J$.

8. Let R be a commutative ring with 1. Assume that R has only one proper maximal ideal. Show that the set $R \smallsetminus R^{\times}$ is the maximal ideal, where $R^{\times}$ is the set of units of R.

9. Let i be the complex number $\sqrt{-1}$.

(a) Show that the ring $\mathbb{Z}[i]$ is a principal ideal domain.

(b) Show that $\mathbb{Z}[i]$ is a unique factorization domain.

(c) What are the units of $\mathbb{Z}[i]$

10. Consider the ring $R = \mathbb{C}[X_1, \ldots, X_n]$. For any ideal $I \subseteq R$, we set

$$\mathcal{V}(I) = \{(x_1, \ldots, x_n) : \forall f \in I \;\; [f(x_1, \ldots, x_n) = 0]\}$$

and for any set $V \subseteq \mathbb{C}^n$ we define

$$\mathcal{I}(V) = \{f \in R : \forall \bar{x} \in V \;\; [f(\bar{x}) = 0]\}.$$

(a) Show that if $I \subseteq J$, then $\mathcal{V}(J) \subseteq \mathcal{V}(I)$.

(b) Show that for any $V \subseteq \mathbb{C}^n$, we have $\mathcal{V}(\mathcal{I}(V)) = V$

(c) Give an example of a maximal ideal I and the set $\mathcal{V}(I)$.

(d) What conclusion can you draw about $\mathcal{V}(I)$ from the assumption that I is a maximal ideal?