MATH 519 QUALIFYING EXAM

SPRING 2019

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let $n \in \mathbb{N}$ be given. Choose n integers $r_1, r_2, \cdots, r_n \in \mathbb{Z}$. Prove that the set $\{a_1 r_1 + a_2 r_2 + \cdots + a_n r_n : a_i \in \mathbb{Z}\}$ is a cyclic group under the usual addition $+$.

2. Let G be a group and let H, K be subgroups of G such that $H < N_G(K)$. Show that $HK < G$ and $K \triangleright HK$. Here HK is the set $\{hk : h \in H, k \in K\}$, which is not necessarily a subgroup of G in general.

3. Let G be a group of oder $5p^4$, where p is a prime (possibly $p = 5$). Show that G is not simple.

4. Let $\phi : G \to H$ be a homomorphism of **finite** groups. Show that $\ker \phi$ contains any subgroup $N < G$ such that $|N|$ is relatively prime to $|H|$.

5. Classify all groups of order 2019. Note that $2019 = 3 \cdot 673$ and 673 is prime.

6. (a) Prove that the centralizer of $\sigma = (12)(34)$ in A_7 has 24 elements.
   (b) Compute the size of the conjugacy class of $\sigma = (12)(34)$ in A_7.

7. Show that the factor ring $\mathbb{Z}[x]/(x^4 + 2x + 3)$ is an integral domain.

8. For any prime p, consider the ring

$$\mathbb{Z}[\sqrt{-p}] := \{a + b\sqrt{-p} : a, b \in \mathbb{Z}\}.$$

Show that the set of all units in $\mathbb{Z}[\sqrt{-p}]$ equals $\{\pm 1\}$.

9. Assume R is a commutative ring with $1 \neq 0$. We say that R is a **Noetherian ring** if it satisfies the ascending chain condition (ACC) on ideals; that is, given any chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

there exists an m such that $I_n = I_m$ for all $n \geq m$. We say that R is an **Artinian ring** if it satisfies the descending chain condition (DCC) on ideals; that is, given any chain of ideals

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

there exists an m such that $I_n = I_m$ for all $n \geq m$.

(a) Show that every finite commutative ring with $1 \neq 0$ is both Artinian and Noetherian.

(b) Is $\mathbb{Z}$ Artinian? Is it Noetherian? Justify your answer.

(c) Give an example of a commutative ring with $1 \neq 0$ that is not Noetherian.

10. Suppose an ideal I in a commutative ring R with $1 \neq 0$ is maximal among all ideals that are not principal. Prove I is a prime ideal.