Math 519 Qualifying Exam

Fall 2020

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let G be a simple group having a non-trivial proper subgroup of index k. Show that G is isomorphic to a subgroup of A_k, the alternating group on k letters.

2. Let G be a finite group with a prime number p dividing $|G|$. Let P be a Sylow p-subgroup of G. Suppose $x, y \in Z_G(P)$, the centralizer of P in G, are conjugate in G. Show that x and y are conjugate in $N_G(P)$, the normalizer of P in G.

   [Note: "x and y are conjugate in a group H" means "$y = gxg^{-1}$ for some $g \in H$."]

3. Let p be a prime number. Consider the group, denoted by $GL_2(p)$, consisting of 2×2 invertible matrices with entries in $\mathbb{Z}/p\mathbb{Z}$ under the matrix multiplication. Note that $|GL_2(p)| = p(p+1)(p-1)^2$. Show that any element of order p is of the form

$$g \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} g^{-1}$$

   for some $g \in GL_2(p)$.

4. (a) Prove that the centralizer of $\sigma = (12)(34)$ in A_6 has 8 elements.

   (b) Compute the size of the conjugacy class of $\sigma = (12)(34)$ in A_6.

5. Let p be a prime and $n \in \mathbb{N}$. The elementary abelian group of order p^n is the group $Z_p \times \cdots \times Z_p$ (n factors). Let A be a finite abelian group. Define

$$A^p = \{a^p \mid a \in A\} \quad \text{and} \quad A_p = \{x \in A \mid x^p = 1\}.$$

   Then A^p and A_p are respectively the image and kernel of the map $\varphi : A \to A$, $\varphi(x) = x^p$.

   a) Prove that $A/A_p \cong A^p$.

   b) Prove that $A/A^p \cong A_p$. (Hint. Show that they are both elementary abelian and they have the same order.)

6. Prove or disprove: If G is a group generated by two finite subgroups H and K, then G is also finite.

7. Let p be a prime number. For any integers $a, b \in \mathbb{Z}$ with $a^2+2b^2 = p$, show that $a+b\sqrt{-2} \in \mathbb{Z}[\sqrt{-2}]$ is a prime element in $\mathbb{Z}[\sqrt{-2}]$.

   [Note: You may use $\mathbb{Z}[\sqrt{-2}]$ is UFD.]

8. For any prime p, consider the following ideal in the ring $\mathbb{Z}[x]$

$$p\mathbb{Z}[x] := \{pf(x) : f(x) \in \mathbb{Z}[x]\}.$$

   Show that $p\mathbb{Z}[x]$ is prime.

9. Let R be a commutative ring with 1, and let I be an ideal of the polynomial ring $R[x]$. Suppose that the lowest degree of a nonzero element of I is n and that I contains a monic polynomial of degree n. Prove that I is a principal ideal.

10. Let $I_1, I_2, \ldots, I_n$ be ideals in a commutative ring R. Suppose

$$I_1 \cap I_2 \cap \cdots \cap I_n = P$$

   is a prime ideal. Prove that $P = I_k$, for some k.