Math 519 Qualifying Exam

Spring 2021

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let p be a prime number. **Show** that any group G of order $519 \cdot p$ is solvable. (Note that the prime factorization of 519 is $3 \cdot 173$. Recall that a group G is *solvable* if there is a chain of subgroups $\{1\} = G_0 \triangleleft G_1 \triangleleft G_2 \triangleleft \cdots \triangleleft G_s = G$ such that G_i is normal in G_{i+1} and G_{i+1}/G_i is abelian, $i = 0, 1, \ldots, s-1$.)

2. Let G, H be finite groups whose orders are relatively prime. **Show** that there is no non-trivial group homomorphism between G and H.

3. Let p be a prime. **Classify** all groups of order $2p$, up to isomorphism.

4. Let G be an Abelian group. We say that $g \in G$ is *divisible* if and only if for every $n \in \mathbb{N}$ there is $h \in G$ such that $nh = g$ (where nh represents the n-fold sum of h). **Show** that any Abelian group G is isomorphic to a direct sum $G \simeq G_1 \oplus G_2$ where every element of G_1 is divisible and no nonidentity element of G_2 is divisible.

5. Let G be a group and $H = \langle \{aba^{-1}b^{-1} : a, b \in G\} \rangle$.
   (a) **Show** that H is normal in G.
   (b) **Show** that any homomorphism of G into an Abelian group factors through G/H.

6. **Prove** that S_5 is not solvable. (Note: the definition of a solvable group is recalled in #1 above.)

7. Consider the fact that any prime ideal of $\mathbb{Z}[x]$ is of the form: (0), (p) with prime number p, $(f(x))$ with irreducible polynomial $f(x) \in \mathbb{Z}[x]$, or $(p, (f(x))$ with prime number p and polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(x)$ modulo p is irreducible (i.e., the image of $f(x)$ via the projection $\mathbb{Z}[x] \to \mathbb{Z}_p[x]$ is an irreducible polynomial in $\mathbb{Z}_p[x]$). **Show** that $\mathbb{Z}[x]/I$ for any ideal I in $\mathbb{Z}[x]$ is a field if and only if I must be of the form $(p, (f(x))$ with prime number p and polynomial $f(x) \in \mathbb{Z}[x]$ such that $f(x)$ modulo p is irreducible.

8. Consider an integral domain $\mathbb{Z}[\sqrt{10}] := \{a + b\sqrt{10} : a, b \in \mathbb{Z}\}$. **Prove or Disprove** that 3 is an irreducible element in $\mathbb{Z}[\sqrt{10}]$.

9. In a ring R, an element a is said to be *nilpotent* if and only if $a^n = 0$ for some $n \in \mathbb{N}$. **Prove** that the following are equivalent:

   (a) R has no nonzero nilpotent elements.

   (b) For all $a \in R$, if $a^2 = 0$ then $a = 0$.

10. Assume that R is a unique factorization domain. Let

$$f = x^3 + x^2y^2 + x^3y + x \in R[x, y].$$

Recall that the content of a polynomial is the greatest common divisor of the coefficients, and that a polynomial is called *primitive* if and only if its content is a unit. **Prove** the following:

   (a) x is irreducible in $R[x]$.

   (b) f, considered as an element of $(R[y])\,[x]$, is primitive.