MATH 520 QUALIFYING EXAM

SPRING, 2005

Directions. Answer 8 of the following 10 questions. Begin each question on a fresh sheet of paper. Hand in only the 8 questions you wish to have graded.

1. Let $\alpha = \sqrt{3} + \sqrt[3]{2}$. Show $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{3}, \sqrt[3]{2})$.

2. Let a, b and c be the roots of $x^3 - 3x + 1$ in $\mathbb{C}$.

(a) Express b and c in terms of a. You may use the discriminant computation

$$(a-b)(b-c)(c-a) = 9.$$

(b) Show $\mathbb{Q}(a)$ is a normal extension of $\mathbb{Q}$.

3. Show $x^4 + x + 1$ is irreducible over $\mathbb{Z}_7$.

4. Suppose $[F(\alpha) : F]$ is odd. Show $F(\alpha^2) = F(\alpha)$.

5. Suppose K is the splitting field of $f(x) \in \mathbb{Q}[x]$ and that $[K : \mathbb{Q}]$ is odd. Show that all of the roots of $f(x)$ are real.

6. Suppose $\mathrm{char}(F) \neq 2$ and $K = F(\sqrt{d})$ is a quadratic extension of F. Consider

$$\begin{aligned} N : K^* &\to F^*/F^{*2} \qquad \text{given by} \\ \alpha &\mapsto N_{K/F}(\alpha)F^{*2}. \end{aligned}$$

Show that $\ker(N) = F^* \cdot K^{*2}$.

7. Let α be a root of an irreducible polynomial of degree d over $\mathbb{Z}_7$. Suppose

$$\alpha^{7^n} = \frac{\alpha + 4}{\alpha + 1}.$$

Prove that $d | 3n$.

8. Suppose there are two finite extensions $F \subset E, F(\alpha)$ with E/F Galois and $F(\alpha) \cap E = F$. Let $f(x)$ be the minimal polynomial of α over F. Show that $f(x)$ is irreducible over E.

9. Let x and y be indeterminants. Let p be a prime and set

$$K = \mathbb{Z}_p(x^{1/p}, y^{1/p}) \qquad F = \mathbb{Z}_p(x, y).$$

(a) Show $[K : F] = p^2$.
(b) Show that $\alpha^p \in F$ for every $\alpha \in K$.
(c) Show that $K \neq F(\alpha)$ for any α.

10. Determine the splitting field $\mathbb{E}$ of $g(x) = x^p - a \in \mathbb{Z}[x]$, where p is a prime and a is not a pth power. Further, determine, $[\mathbb{E} : \mathbb{Q}]$.