Mathematics 569 September 1998

Ph. D. Qualifying Examination

For this exam you may use the textbook, N. N. Lebedev, *Special Functions and Their Applications*, as a reference during the exam, but no other outside material may be used.

1. Compute

$$\psi(1/3).$$

2. Show that

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh(\pi y)}.$$

3. Compute

$$\int_0^{+\infty} \frac{dt}{(1+t)^2\sqrt{1+1/t}}.$$

4. Define the Gegenbauer polynomials by the generating function

$$\left(1-2xt+t^2\right)^{-\gamma} = \sum_{n=0}^{+\infty} C_n^{\gamma}(x)t^n,$$

where $\gamma > 0, -1 \le x \le 1, |t| < 1$.

(a) Find the three-term recursion relation satisfied by the $C_n^{\gamma}(x)$.

(b) Find the differential equation satisfied by the $C_n^{\gamma}(x)$.

5. Let $\{L_n^{\alpha}(x)\}$ be the set of Laguerre polynomials. Show that, for $\mathrm{Re}(\alpha), \mathrm{Re}(\gamma) > 0$, we have

$$\int_0^{+\infty} x^{\alpha} e^{-\mu x} L_n^{\alpha}(x)dx = \frac{\Gamma(n+\alpha+1)}{\mu^{\alpha+1} n!}\left(1-\frac{1}{\mu}\right)^n.$$

6. Show that, if $\mathrm{Re}(\mu+\nu)>-1$, then

$$J_\mu(z)J_\nu(z)=\frac{2}{\pi}\int_0^{\pi/2}J_{\mu+\nu}(2z\cos\theta)\cos((\mu-\nu)\theta)d\theta.$$

7. Show that if $\mathrm{Re}(\nu)>-1$, then

$$\int_0^z J_\nu(t)dt=2\sum_{n=0}^{+\infty}J_{\nu+2k+1}(z).$$

8. Show that

$$I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=\frac{1}{z}.$$

(Hint: think Wronskian.)

9. Show that

$$2^m\frac{d^m}{dz^m}(J_n(z))=\sum_{k=0}^{m}\binom{m}{k}(-1)^{m-k}J_{n+m-2k}(z).$$

10. Show that if n is a nonnegative integer, then

$$Q_n(z)=\tfrac{1}{2}P_n(z)\log\frac{z+1}{z-1}-f_{n-1}(z),$$

where $P_n(z)$ is the nth Legendre polynomial and $f_{n\text{-}1}(z)$ is a certain polynomial.

11. Let, for $0\le k<1$,

$$K(k)=\int_0^{\pi/2}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}$$

be the elliptic integral of the first kind. Show that

$$P_{-1/2}(\cosh\alpha)=\frac{2}{\pi}\mathrm{sech}(\alpha/2)K(\tanh(\alpha/2)).$$